CHAPTER 1
INTRODUCTION

1.1. Review Of Literature

Incline is an algebraic structure and is a special type of a semiring. In an incline $(£, +, \cdot)$ with the order relation "$\leq$" defined on $£$ as $x \leq y$ if and only if $x+y=y$ for $x,y \varepsilon £$, the incline axioms, that is, $x+xy = x$ and $y+xy=y$, imply that $xy \leq x$ and $xy \leq y$. Thus, inclines are additively idempotent semirings in which products are less than (or) equal to either factor. Further, Incline algebra is a generalization of Fuzzy algebra which is a generalization of Boolean algebra.

The concept of slope was introduced by Cao [7] and later Cao, Kim and Roush [11] renamed it as incline. The notion of inclines and their applications are described comprehensively in Cao, Kim and Roush [11]. Kim and Roush [27] have surveyed and outlined algebraic properties of inclines and incline matrices. In [29, 30], the authors have discussed the inclines of algebraic structures. Various families of finite inclines and of finite inclines representable by Boolean matrices have been discussed in [31].

Inclines and incline matrices have good vistas of applications in diverse areas such as automata theory, graph theory, medical diagnosis, informational systems, complex systems modeling, decision making theory, dynamical programming, control theory, clustering and so on. Incline algebra and incline matrix theory have been extensively studied by many researchers [1, 2, 6, 8, 9, 10, 16, 17].

The power sequence of the matrices over an incline, that is, indices, periods, convergence and orbits are discussed in detail by Han and Li [18]. Han and Li [21], studied the eigenvectors of the incline matrix which has index. The maximum eigenvector of a square matrix with a given associated eigenvalue over a complete and completely distributive lattice was determined by Tan [46].

In [15,19], some invertible conditions of a matrix over a commutative incline £ with additive identity "$0_£$" and multiplicative identity "$1_£$" are obtained and also it is established that an incline £ is an integral incline if and only if the group of all invertible matrices coincides with the group of all permutation matrices. Invertible matrices over an incline and Cramer's rule are investigated in [20]. Invertible conditions for matrices over distributive lattice are discussed in [45, 49, 50]. An incline matrix, that is, matrix over an incline $A \in £_n$ is invertible if and only if there exists $B \in £_n$ such that $AB = BA = (I_£)_n$, the identity matrix of order n. A matrix $A \in £_{mn}$, the set of $m \times n$ matrices over an incline £ is regular if and only if there exists $X \in £_{nm}$ such that $AXA = A$ and X is called a g-inverse of A. It is clear that, every invertible matrix is regular. Thus regular matrices are a generalization of invertible matrices.

Von Neumann [48] has introduced the concept of regular elements in a ring. An element a is called regular if a solution exists for the equation $axa = a$ and such solution is called a generalized inverse of a. A ring R is regular if and only if every element of R is regular. In [22], Hartwig has studied on existence and construction of various g - inverses associated with an element in a *-regular ring, that is, regular ring with an anti - automorphism and developed a technique for computing g-inverses mainly by using star cancellation law. Recently in [40], it is proved that an element in an incline is regular if and only if it is idempotent, further some characterization of regular elements in an incline are obtained by using the incline axioms, without using star cancellation law as in [22] and exhibited that every commutative regular incline is a distributive lattice.

It is well known that, a ring R is regular, if and only if R_{mn}, the set of all $m \times n$ matrices over R is regular. However, this fails for matrices over a Fuzzy algebra and therefore for matrices over an incline. Further, incline matrices need not form an incline, since, the incline axiom $AB \leq A$ (or) $AB \leq B$ need not hold. For $A = (a_{ij})$ and $B = (b_{ij})$, $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all i and j.

Hence, A $\leq B \Leftrightarrow A+B=B$. However, under the entrywise (Schur (or) Hadamard) product the set of $m \times n$ matrices over £ forms an incline. In [12], it is shown that R_n, the set of $n \times n$ matrices over R is regular semiring then R is regular semiring but the converse need not be true for $n = 2$.

In general, for an incline matrix, the row and column ranks are not equal as in the case of complex matrices. In [27], it is proved that for a regular matrix over a finite incline whose idempotent elements are linearly ordered, the row and column ranks are equal. Kim and Roush [28] have studied the existence and construction of various g-inverses for matrices over the Fuzzy algebra analogous to that for complex matrices [5]. The concept of full rank factorization was introduced by Meenakshi and Sriram [39] for fuzzy matrices, that is, matrices over the Fuzzy algebra under max-min composition. For more details on Fuzzy matrices, refer [36].

Sanchez [42] initiated the study on fuzzy relational equations of the form $xA=b$, based on the max-min composition. A method of determining minimum solutions of fuzzy relational equations are provided in [23]. Cho [13], has proved that $xA=b$ is consistent when A is regular and $x=Xb$ where X is a g-inverse of A is a solution of $xA=b$. In [43,44], Sanchez has obtained the resolution of eigen fuzzy sets equations.

The notion of commitment scheme is at the heart of most of the constructions of modern Cryptography protocols. Commitment protocols were introduced by Blum [33]. Fuzzy commitment scheme was introduced by Juels and Martin [25]. McEliece proposed the first public key cryptosystem based on algebraic coding theory in [34,35]. Later, his technique was named as McEliece Scheme. In [3,4,14,26], the authors have discussed Fuzzy Commitment Scheme in Cryptography to determine the methods for encryption and decryption by using the McEliece scheme. In [24], Jordan discussed the public - key cryptosystem by using Goppa codes. The equivalence of deterministic finite state automata (DFSA) and the conversion of a non deterministic finite state automata (NDFSA) to an equivalent DFSA play an important role in Automata Theory (refer [32, 47]).

In the present investigation, we have discussed the regularity of matrices over an incline *(£, ≤)* with least and greatest element under the order relation "≤", as an extension of the results on regular elements of an incline found in [40], as a generalization of the results on fuzzy matrices available in the literature [28, 38, 39] and as a development of the results of an invertible incline matrices discussed in [15, 19, 20]. We have determined the condition for the consistency of incline relational equation as an extension of the Sanchez's method [42] and highlighted the role of incline matrices in Automata Theory and in Cryptography.

1.2. Notations

$£$	:	Incline
$(£, \leq)$	:	Incline with order relation "$\leq$"
$(£, \leq, \{0,1\})$	:	Incline with the least element "0" and the greatest element '1' under the order relation "$\leq$"
I_n	:	the diagonal matrix of order n, whose entries are the greatest element '1'.
$1_£$	:	Multiplicative identity on £, that is $x \cdot 1_£ = 1_£ \cdot x = x$, for all $x \varepsilon £$
$(I_£)_n$	:	the diagonal matrix of order n, whose diagonal entries are "$1_£$"
$0_£$	:	Additive identity on £, that is, $x + 0_£ = x = 0_£ + x$, for all $x \varepsilon £$
$£_{mn}$	:	the set of all $m \times n$ matrices over $£$
$£_n$	:	the set of all $n \times n$ matrices over $£$
$£^m$	:	the set of all $1 \times m$ vectors over $£$
DL	:	the set of all idempotent elements in $£$ $= \{x \varepsilon £ / x^2 = x\}$
DL_{mn}	:	the set of all $m \times n$ matrices over DL
DL_n	:	the set of all $n \times n$ matrices over DL
DL^n	:	the set of all $1 \times n$ vectors over DL
$<s>$	:	the span of a set S
ß	:	basis

$\mathcal{P}(D)$	:	power set of D
N_r	:	the set of all positive integers 1 to r

For $A \varepsilon \pounds_{mn}$,

A^T	:	the transpose of A
A_{i*}	:	the ith row of A
A_{*j}	:	the jth column of A
A^-	:	the g-inverse of A, that is, a solution of the matrix equation $AYA = A$.
$A^\dagger$	:	the Moore - Penrose inverse of A
$A\{1\}$	:	the set of all g - inverses of A
$\mathcal{R}(A)$	:	the row space of $A = \{y=xA,$ for $x\varepsilon \pounds^m\}$
$\mathcal{C}(A)$	:	the column space of $A = \{y=xA^T,$ for $x \ \varepsilon \pounds^n\}$
$\rho_r(A)$	:	the row rank of A
$\rho_c(A)$	:	the column rank of A
$\rho(A)$	:	the rank of A
$\Omega(A,b)$	:	the solution set of $xA = b,$ where $A \varepsilon \pounds_{mn}$ and $b \varepsilon \pounds^n$.
$A \odot B$	:	entrywise (Schur (or) Hadamard) product of A and B that is, $A \odot B = (a_{ij} \cdot b_{ij}) \varepsilon \pounds_{mn}$ for $A = (a_{ij}) \varepsilon \pounds_{mn}$ and $B = (b_{ij}) \varepsilon \pounds_{mn}$.
$A^{ⓝ}$	:	the Schur product of $A = (a_{ij}) \varepsilon \pounds_{mn}$ n times.

Basic Definitions and Preliminaries

In this section, basic definitions and results required on elements of a semiring R are presented. Let R be any semiring with the least element denoted as *‘0’* and the greatest element denoted as *‘1’* with respect to an order relation “$\leq$”.

Definition 1.2.1

A subspace of V^n is a subset W of V^n such that $0 \varepsilon W$ and for $v,w \varepsilon W$, we have $v+w \varepsilon W$.

Definition 1.2.2

A set S of vectors over a semiring R is independent if and only if each element of S is not a linear combination of other elements of S, that is, no element $v \varepsilon S$ is a linear combination of elements in $S| \{v\}$.

If $v \varepsilon S$ is a linear combination of elements of $S| \{v\}$, then S is said to be dependent.

Definition 1.2.3

A basis for a subspace W of V^n is a smallest linearly independent set S of vectors such that $<S> = W$, where $<S>$ is the space spanned by the set S.

Definition 1.2.4

The row space $\mathcal{R}(A)$ of an $m \times n$ matrix A is the subspace of V^n generated by its rows. The row rank $\rho_r (A)$ is the smallest possible size of a spanning set for the row space. The column space $\mathcal{C}(A)$ and column rank $\rho_c(A)$ are defined in dual fashion. If row rank equals column rank, then it is called the rank of A and is denoted as $\rho(A)$.

Definition 1.2.5

For $n \times n$ matrices X,Y over a commutative semiring R we have the following :

(i) $A \mathcal{R} B \Leftrightarrow \mathcal{C}(A) = \mathcal{C}(B)$

(ii) $A \mathcal{L} B \Leftrightarrow \mathcal{R}(A) = \mathcal{R}(B)$

(iii) $A \mathcal{H} B \Leftrightarrow \mathcal{C}(A) = \mathcal{C}(B)$ and $\mathcal{R}(A) = \mathcal{R}(B)$

Definition 1.2.6

For a matrix A in semiring R. Consider the following four equations:

(1) $AXA = A$ (2) $XAX = X$ (3) $(AX)^T = AX$

(4) $(XA)^T = XA$

Here, A^T is the transpose of A. X is said to be a λ - inverse of A and $X \varepsilon A\{\lambda\}$ if X satisfies λ-equation, where λ is a subset of $\{1,2,3,4\}$. In particular, if $\lambda = \{1,2,3,4\}$ then X is called the Moore - Penrose inverse of A and is denoted as $A^{\dagger}$.

For $\lambda = \{1\}$, A is regular, $X \varepsilon A\{1\}$ is called the g-inverse of A and $A\{1\}$ denotes the set of all g-inverses of A.

Remark 1.2.7

From the Definition (1.2.6), $X \varepsilon A \{1,3\} \Leftrightarrow X^T \varepsilon A^T\{1,4\}$

Lemma 1.2.8

Let R be any semiring. For $A,B \varepsilon R_{mn}$, we have the following :

(i) $\mathcal{R}(B) \subseteq \mathcal{R}(A) \Leftrightarrow B = XA$ for some $X \varepsilon \pounds_m$.

(ii) $\mathcal{C}(B) \subseteq \mathcal{C}(A) \Leftrightarrow B=AY$ for some $Y \varepsilon \pounds_n$.

Lemma 1.2.9

Let R be any semiring. For $A,B \varepsilon R_{mn}$, we have the following :

(i) $\mathcal{R}(AB) \subseteq \mathcal{R}(A)\, B \subseteq \mathcal{R}(B)$

(ii) $\mathcal{C}(AB) \subseteq \mathcal{C}(A)$

Lemma 1.2.10

Let R be any semiring and $A,B \varepsilon R_{mn}$. If A is a regular matrix then the following hold:

(i) $\mathcal{R}(B) \subseteq \mathcal{R}(A) \Leftrightarrow B = BA^{-}A$ for each $A^{-} \varepsilon A\{1\}$

(ii) $\mathcal{C}(B) \subseteq \mathcal{C}(A) \Leftrightarrow B = AA^{-}B$ for each $A^{-} \varepsilon A\{1\}$

Definition1.2.11

A nonempty set £ with two binary operations + and · is called an incline if it satisfy the following conditions (We usually suppress the 'dot' in $x{\cdot}y$ and write as xy)

(i) $(£, +)$ is a semilattice.

(ii) $(£,\cdot)$ is a semigroup.

(iii) $x(y+z) = xy+xz$ for all $x,y,z \varepsilon £$.

(iv) $x+xy = x$ and $y + xy = y$ for all $x,y \varepsilon £$.

Since an incline £ is a special type of a semiring and a generalization of the Boolean and Fuzzy algebra, the above definitions and results on a semiring remain valid for matrices over an incline £.

Definition 1.2.12

For $x,y \varepsilon £$, the order relation "≤" is defined as $x \leq y \Leftrightarrow x+y = y$. From the incline axiom (iv), the order relation "≤" has the following properties;

$$x+y \geq x \text{ and } x+y \geq y \text{ for } x,y \varepsilon £ \qquad \rightarrow (1.2.1)$$

$$xy \leq x \text{ and } xy \leq y \text{ for } x,y \varepsilon £ \qquad \rightarrow (1.2.2)$$

which are called as Incline Properties.

Definition 1.2.13

Let $A = (a_{ij}) \varepsilon £_{mn}$ and $B = (b_{ij}) \varepsilon £_{mn}$. A is said to dominate B, that is, $A \geq B$ if and only if $a_{ij} \geq b_{ij}$ for all i and j.

Proposition 1.2.14

For any $A = (a_{ij})\ \varepsilon\ £_{mn}$ and $B = (b_{ij})\ \varepsilon\ £_{mn}$, for all i and j.

Then $A \leq B \quad \Leftrightarrow a_{ij} \leq b_{ij}$

$\Leftrightarrow A+B = B.$

Definition 1.2.15

The multiplicative identity $1_£$ on £, is defined as $x1_£ = 1_£\ x = x,$ for all $x\ \varepsilon £.$

The additive identity $0_£$ on £, is defined as $x + 0_£ = x = 0_£ + x,$ for all $x\ \varepsilon\ £.$

Definition 1.2.16 [19]

An incline £ with additive identity $0_£$ and multiplicative identity $1_£$ is called an integral incline if there do not exist non zero elements x and y in £ such that $xy = 0_£$ and $x+y = 1_£.$

Definition 1.2.17

A set S is said to be a comparable subset of an incline £ if and only if the elements of S are comparable, that is, either $x \leq y$ (or) $y \leq x$ for each $x,y\ \varepsilon\ S.$

Definition 1.2.18

$a\ \varepsilon\ £$ is said to be regular if there exists an element $x\ \varepsilon\ £$ such that $axa = a.$ Then x is called a g-inverse of a and $a\{1\}$ denotes the set of all g-inverses of a.

Proposition 1.2.19 [40]

An element $a\ \varepsilon\ £$ is regular if and only if a is idempotent if and only if $a^2 = a.$

Proposition 1.2.20

An incline £ is regular if and only if each element of £ is regular, if and only if $DL = £.$

Proposition 1.2.21 [40]

A commutative incline $£$ is regular if and only if $£$ is a distributive lattice.

Lemma 1.2.22[40]

Let $a \varepsilon £$ be regular. Then $a = ax = xa$ for all $x \varepsilon a\{1\}$.

Proposition 1.2.23 [40]

For a regular element $a \varepsilon £$, a is the smallest g - inverse of a, that is, $a \leq x$ for all $x \varepsilon a\{1\}$ and $a\{1,2\} = a$.

Definition 1.2.24

A basis C over the incline $£$ is a standard basis if and only if whenever $c_i = \Sigma a_{ij} c_j$ for c_i, $c_j \varepsilon C$ and $a_{ij} \varepsilon £$ then $a_{ii} c_i = c_i$.

Theorem 1.2.25 (p.38,[11])

Every subspace of a finite incline whose idempotent elements are linearly ordered has a unique standard basis.

Definition 1.2.26

A subspace W of $£^n$ is a retract of $£^n$ if there exists an idempotent matrix $X \varepsilon £_n$ such that $(£^n) X = W$.

Proposition 1.2.27

For an idempotent matrix E, $\mathcal{R}(E) = (£^n) E$.

Definition 1.2.28

Let $A = (a_{ik}) \varepsilon £_{mn}$ and $B = (b_{kj}) \varepsilon £_{nl}$ then the product $AB = S$ is defined as $AB = \sum_{k=1}^{n} a_{ik} b_{kj} = s_{ij}$ for all i and j, where 'Σ' denotes the addition operation on $£$.

Definition 1.2.29 [19]

A matrix $A \varepsilon £_n$ is invertible $\Leftrightarrow AX = XA = (I_£)_n$ for some $X \varepsilon £_n$.

Definition 1.2.30 [19]

An incline $\pounds$ with the additive identity "$0_{\pounds}$" and a multiplicative identity "$1_{\pounds}$", a matrix $P \varepsilon \pounds_n$ is called a permutation matrix if it has exactly one entry equals "$1_{\pounds}$" in each row and each column of P and all other entries are "$0_{\pounds}$".

Theorem 1.2.31 [11]

Let $\pounds$ be an incline whose idempotent elements are linearly ordered. Let $A \varepsilon \pounds_{mn}$ be a regular matrix, whose non zero rows form a standard basis. Then A has a g-inverse which is a permutation matrix.

Definition 1.2.32 [18]

For a matrix $A \varepsilon \pounds_{\text{n}}$ then we have the following:

(i) Reflexive $A \geq I_n$

(ii) Symmetric $A = A^T$

(iii) Idempotent $A = A^2$

(iv) Transitive $A^2 \leq A$.

1.3. Motivation

- Regular rings are important in many branches of mathematics, and especially in matrix theory, since the regularity condition is a linear condition that solves linear equations and takes the place of canonical decompositions. This motivated us to study on regular matrices over an incline as a development of the work of Cao, Kim and Roush [11,18, 19, 20, 27] and as a generalization of the results in Fuzzy algebra, in particular, fuzzy matrices [28,36].

- In [40], it is proved that for a regular element a in an incline $a\{1,2\}=a$, that is, a is the only solution for the equations $axa = a$ and $xax = x$. However, this need not be the case for matrices over an incline. This motivated us to study on various g-inverses of a matrix over an incline as a generalization of the results found in [13, 28, 36]. We have initiated with a study on regularity of matrices over an incline analogous to that of complex matrices studied and developed by C.R.Rao [41].

- In our study on matrices over an incline, we are confined to the following three main classifications of an incline.

(i) Incline
- Regular
- Non regular

(ii) Incline
- Commutative
- Non Commutative

(iii) Incline
- Whose elements are all linearly ordered
- Whose elements are all not linearly ordered

In general, there is no inter-relation between these three classifications. We have observed that a regular incline whose elements are all linearly ordered is commutative and hence a distributive lattice.

- We exhibit that the greatest element coincides with the multiplicative identity for a regular incline, that is, for an incline whose elements are all idempotent. We observe that if the multiplicative identity of an incline exists, then it coincides with the greatest element. This motivated us to study on regularity of matrices over a non regular incline without multiplicative identity and with the greatest element.

1.4. Summary of the results

A short account of the results obtained in this thesis is given.

Vector spaces over an incline

We prove that every finite subspace generated by the linearly ordered elements in an incline has a unique standard basis. This leads to the existence of a unique standard basis for every finite subspace of a regular incline whose elements are all linearly ordered and thereby we disprove the result of Cao [27], that is, "Every subspace of a finite incline whose idempotent elements are linearly ordered has a unique standard basis". We have introduced the concept of space factorization for matrices in an incline. We have discussed the full row space factorization, the full column space factorization and full space factorization as a generalization of the results on fuzzy matrices found in [39], which forms a part of the author's paper appeared in "The International Journal of Computational Cognition". We exhibit that, in the incline *£*, the greatest element *'1'* is the multiplicative identity for the elements of *DL*. In particular, for a regular incline *£, DL* coincides with *£* and *'1'* coincides with the multiplicative identity "$1_{£}$" of *£*.We have discussed the invertibility of matrices over *DL*, the set of idempotent elements, in an incline *£* and for matrices over an integral incline. In particular, for matrices over special types of inclines such as regular incline whose elements are all linearly ordered, for distributive lattice whose elements are all linearly ordered, our results reduce to the results established in [11, 19, 20]. We have discussed the structure of Range symmetric incline matrix, a special type of incline matrix, which is a generalization of invertible matrix over an incline.

Regular matrices over an incline

We observe that a regular incline £ whose elements are all linearly ordered is commutative and the operations on £ reduces to the max-min composition. Equivalent conditions for regularity of a matrix over an incline with linearly ordered idempotent elements are obtained, which include as special cases for matrices over a regular incline whose elements are all linearly ordered and for matrices over a distributive lattice whose elements are all linearly ordered [19,27] and as a generalization of the results found in [13,28]. The contents form a part of the material of the author's paper appeared in "The International Journal of Computational Cognition". We have discussed the regularity of matrices over an incline in terms of full row space, full column space factorization and studied the properties of g-inverses of matrix over *DL*. An algorithm is given in [27] to determine the regularity and finding g-inverses for matrices over an incline in which idempotent elements are linearly ordered. In step 1 of the algorithm, one has to check the maximal element row by row, which cannot be applied when elements are not comparable. We provide an algorithm for matrices over a regular incline £ whose elements are not linearly ordered and illustrated with suitable examples.

Generalized Inverses of Incline Matrices

We have discussed the existence and construction of various generalized inverses associated with a matrix over an incline whose elements are all idempotent as a generalization of the results available in the literature [28]. A formula is provided to compute the Moore - Penrose inverse of an incline matrix. We have determined the characterization of various g-inverses associated with a regular matrix over an incline. These results are based on the author's paper appeared in the journal of "Advances in Algebra" We have discussed the characterization of range symmetric matrix involving various g-inverses.

Incline Relational Equations with Applications

We have discussed the consistency of incline relational equations, that is, equations of the form $xA=b$, where A is a matrix and b is a vector over an incline. We have determined the existence of the maximum solution of $xA=b$ under the condition that each column of A is comparable with the corresponding component of the vector b in £. This leads to the structure of the solution set $\Omega(A,b)$, where A is a matrix over special types of inclines such as incline whose elements are all linearly ordered, incline whose idempotent elements are all linearly ordered, a regular incline whose elements are all linearly ordered and a distributive lattice whose elements are all linearly ordered. This includes the result found in [23, 42] as a special case for fuzzy relational equations. By using the maximum solution of the incline relational equation, we have discussed when a vector can be expressed uniquely as a linear combination of the standard basis vectors. As a special case, we have exhibited that each vector in a regular incline whose elements are all linearly ordered has a unique decomposition as a linear combination of its standard basis vectors, which we call as standard incline linear combination. This is a generalization of standard linear combination of a vector over the max-min Fuzzy algebra [37]. We apply our results on regularity of matrices over an incline in the determination of the equivalence of finite state machines in Automata. We have highlighted an application of incline matrices over the incline £ under the operations addition as supremum and usual multiplication in Cryptography for encryption and decryption based on McEliece scheme [34]. The concept itself is illustrated with the help of a simple situation and the validation of mathematical experimental verification is provided.

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CHAPTER 2

VECTOR SPACES OVER AN INCLINE

In this Chapter, we prove that every finite subspace of DL^n, the space of vectors of idempotent elements over a finite incline £ whose idempotent elements are linearly ordered has a unique standard basis. This leads to the existence of a unique standard basis for every finite subspace of a regular incline whose elements are all linearly ordered and thereby we disprove the result of Cao [11], that is, "Every subspace of a finite incline whose idempotent elements are linearly ordered has a unique standard basis". We have introduced the concept of space factorization for matrices over an incline. We have discussed the full row space factorization, the full column space factorization and the full space factorization as a generalization of the results on fuzzy matrices found in (p.24, [36]). We have obtained the relation between the greatest element *'1'* of an incline and the multiplicative identity "$1_£$" of an incline £. We exhibit that, in an incline £, the greatest element *'1'* is the multiplicative identity of the elements of *DL*. In particular, for a regular incline £, *DL* coincides with £ and therefore *'1'* coincides with the multiplicative identity "$1_£$" of £. On the other hand, for any incline £, that has the multiplicative identity "$1_£$", it coincides with the greatest element *'1'* (refer Example (2.1.8)).we have discussed the invertibility of a matrix over *DL.* We have introduced the concept of range symmetric incline matrix analogous to that for complex matrices (p.163, [5]) and as a generalization of range symmetric fuzzy matrices (p.118, [36]).

2.1. STANDARD BASES

In this section, we prove that every finite subspace of DL^n over a finite incline £ whose idempotent elements are linearly ordered has a unique standard basis. As a special case, we deduce that every finite subspace of a regular incline whose elements are all linearly ordered has a unique standard basis.

Definition 2.1.1

A subincline of an incline £ is a subset closed under the incline operations addition and multiplication.

Definition 2.1.2

A basis C over the incline £ is a standard basis if and only if whenever $c_i = \Sigma a_{ij}c_j$ for c_i, $c_j \varepsilon C$ and $a_{ij} \varepsilon$ £ then $a_{ii}\, c_i = c_i$.

Just for sake of completeness we shall prove the following Theorem for a non commutative incline which has been proved for a commutative incline in (p.36, [11]).

Theorem 2.1.3

In a finite incline, for any basis ß, there exists a standard basis of the same cardinality as ß and same spanning set.

Proof

Let us assume that C is not a standard basis for ß, that is, by Definition (2.1.2), if $x_k = \Sigma\ c_i x_i$ for some k, then $x_k \neq c_k\ x_k$. Hence by Incline Property $c_k x_k < x_k$.

Let C' be the set obtained from C by replacing x_k by $c_k\ x_k$. Then C' has the same cardinality as C, $|C| = |C'|$. Let us define the weight of C as the sum of all the elements of C. Therefore, weight of C' < weight of C.

The set C' spans the same space, since $x_k = \Sigma\ c_i x_i$, there is no other x_j will be dependent. If $c_k\, x_k$ is dependent, then $x_k = \Sigma\ c_i x_i$. If C' is a standard basis, then the result holds. If not, then repeat this process. Hence after a finite number of steps this process will terminate, and thereby ß will have a standard basis.

Lemma 2.1.4

In a vector space over a finite incline, if $x = \Sigma c_i x$, then for some n, $x = \Sigma\, c_i^n x$ where c_i^n is idempotent.

Proof

Since $x = \Sigma\, c_i\, x$, take $c = \sum_{i=1}^{k} c_i$ then by using the incline axiom (iv) of Definition (1.2.11) repeatedly in $x= cx$ we get $x = c^k x = c^{k+1} x$. By using Incline Property (1.2.2), we get $c \geq c^2 \geq c^3 \geq \ldots\ c^n \geq \ldots$ Since £ is a finite incline, c^n is idempotent for some n. →(2.1.1)

Again by using Incline Property (1.2.2), for $c = \sum_{i=1}^{k} c_i$, we get $c \geq c_i$ for each $i=1$ to k. Hence, $c \geq c_i \geq c_i^2 \geq \ldots \geq c_i^m \geq \ldots$, for a finite incline, c_i^{mi} is idempotent for some m_i and for each $i=1$ to k. →(2.1.2)

From (2.1.1) we get $c^n = c^{2n}$ and from (2.1.2), $c_1^{m_1} = c_1^{2m_1}$, $c_2^{m_2} = c_2^{2m_2}, \ldots\ c_k^{m_k} = c_k^{2m_k}$ and $c \geq c_i^{m_i}$ for each $i=1$ to k. Choose the power of c which is the maximum of m_i for each $i=1$ to k.

Hence $x = \Sigma\, c_i^n\, x$ where c_i^n is idempotent for some $n > 1$.

Lemma 2.1.5

Let x be a member of a standard basis of DL^n over a finite incline £ whose idempotent elements are all linearly ordered. If $x = \Sigma\, y_i$ for y_i in the space spanned by the standard basis of a finite subspace of DL^n then $x = y_k$ for some k.

Proof

Let $\{x_1, x_2, \ldots., x_k\}$ be the standard basis of DL^n and $x=x_1$. Write $y_i = \Sigma_j\, c_{ij}\, x_j$. Then $x = \Sigma y_i = \Sigma_j\, (\Sigma_i\, c_{ij}\,)\, x_j$ and by Lemma (2.1.4), we get $x = \Sigma\, c_{i1}^n x$, where c_{i1}^n is idempotent. Since the elements of DL are linearly ordered, let c_{j1}^{n} be the largest. Then $x = c_{j1}^{n}x$. By using Incline Properties (1.2.1) and (1.2.2), $x = \Sigma y_i \Rightarrow x \geq y_i$ and $y_i = \Sigma c_{ij}\, x \Rightarrow x \geq y_j \geq c_{j1}^{n}\, x = x$. Hence, $x{=}y_j$.

Theorem 2.1.6

Let $£$ be an incline whose idempotent elements are all linearly ordered then any finite subspace of DL^n has a unique standard basis.

Proof

Let $\{x_1, x_2,\ldots,x_k\}$ and $\{y_1,y_2, \ldots,y_j\}$ be the distinct standard basis of a finite subspace of DL^n. For any x_i, we have $x_i = \Sigma\, c_{ik}y_k$ for some c_{ij}. Then by Lemma (2.1.5), $x_i = c_{ik}y_k$ for some k. In the same manner, $y_k = d_{kj}\, x_j$ for some j. If $i{\neq}j$, then x_i would be dependent. Hence, $x_i = c_{ik}\, y_k = c_{ik}d_{ki}x_i$ by Lemma (2.1.4), $c_{ik}^{n}d_{kj}^{n}\, x_i = x_i$. This proves the uniqueness of the standard basis.

Remark 2.1.7

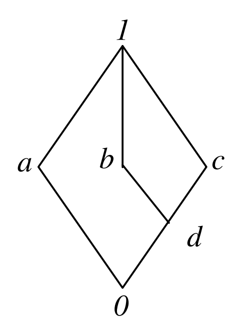
In the above Theorem (2.1.6), the condition that any subspace of DL^n is essential. This is illustrated in the following example:

Example 2.1.8

Let us consider the incline $£ = \{0,a,b,c,d,1\}$, lattice ordered by the following Hasse graph.

Define $£ \times £ \rightarrow £$ as follows

$$xy = \begin{cases} d \text{ if } x,\ y\ \varepsilon\ \{1,b,c,d\} \\ 0 \text{ otherwise} \end{cases}$$



In this finite incline, the only idempotent elements are 0 and d and they are comparable. Since $0 = \alpha d$ and $d = \beta d$ for $\alpha,\beta\ \varepsilon\ £;$ $\{d\}$ is the unique standard basis for $DL = \{0,d\}$. Here $\{d\}$ is not a basis for the subincline $I = \{0,a,b,d,1\}$.For instance, $b\ \varepsilon\ I,\ b \neq \alpha d$ for all $\alpha\ \varepsilon\ £,$ therefore b cannot be expressed as a linear combination of the elements of $£$.

This contradicts the Theorem (1.2.25) of Cao (p.38,[11]).

Now we deduce the result of Theorem (2.1.6) for the following special type of inclines.

Corollary 2.1.9

Every finite subspace of a regular incline whose elements are all linearly ordered has a unique standard basis.

Proof

Since *£* is a regular incline by Proposition (1.2.20) each element of *£* is idempotent, hence DL = *£*. Then the rest follows from Theorem (2.1.6) under the condition that all elements are linearly ordered.

Remark 2.1.10

Since by Proposition (1.2.21), a commutative regular incline is a distributive lattice and DL = *£,* Theorem (2.1.6) reduces to the following:

Corollary 2.1.11

Every finite subspace of a distributive lattice whose elements are all linearly ordered has a unique standard basis.

Remark 2.1.12

For a Fuzzy algebra with support [0,1] under the operation max-min (or min-max) the elements are all idempotent and linearly ordered. Hence Theorem (2.1.6) reduces to the following result due to Kim and Roush [28] (p.8,[36]).

Corollary 2.1.13

Any finitely generated subspace of a Fuzzy algebra has a unique standard basis.

2.2. Rank and space factorization

In Section 2.1, we have established that the cardinality of all bases of a finite dimensional subspace of an incline is independent of the choice of basis, and this leads to the notion of rank of a matrix over a finite incline [27]. In this section, we have discussed the rank factorization of a matrix over an incline and derived some basic properties as a generalization of the results for fuzzy matrices proved in [39] and illustrate with suitable examples. We have introduced the concept of space factorization for matrices over an incline and derived some basic properties and illustrate with suitable examples.

Definition 2.2.1

For $A \varepsilon \pounds_{mn.}$ The row rank $\rho_r(A)$ is the smallest possible size of a spanning set of $\mathcal{R}(A)$. The column space $\mathcal{C}(A)$ and column rank $\rho_c(A)$ are defined in dual fashion. Since $\mathcal{C}(A) = \mathcal{R}(A^T)$, $\rho_c(A) = \rho_r(A^T)$.

Example 2.2.2

Let us consider the incline $\pounds = \{[0,1], sup\ (x,y), inf\ (x,y)\}$.

Let $A = \begin{pmatrix} 1 & 0.4 \\ 0.4 & 0 \end{pmatrix} \varepsilon\ \pounds_2$

Any element (x,y) in $\mathcal{R}(A)$ is of the form

$$\begin{aligned} (x,y) &= \alpha\ (1,0.4) + \beta\ (0.4,0) \text{ for } \alpha, \beta\ \varepsilon\ \pounds \\ &= (\alpha,\alpha(0.4)) + (\beta\ (0.4),0) \\ x &= sup\ \{\alpha,\beta(0.4)\} \\ y &= \alpha\ (0.4) = inf\ (\alpha,\ 0.4) \end{aligned}$$

If $\alpha \geq 0.4$ then $y = 0.4$ and $x = \alpha$.

If $\alpha \leq 0.4$ then $y = \alpha$ and $x \leq 0.4$.

Therefore, $0 \leq y \leq 0.4$

Hence $\mathcal{R}(A) = \{(x,y) | 0 \leq y \leq x \leq 0.4\} \cup \{(x,y) | 0.4 = y \leq x \leq 1\}$

Here, $A = A^T$. Hence $\mathcal{C}(A) = \mathcal{R}(A)$ and

$$\rho_r(A) = \rho_c(A) = 2.$$

Definition 2.2.3

Let $A \varepsilon \pounds_{mn}$. The factor rank $\rho_f\,(A)$ is the smallest integer t such that $A = BC$ where $B \varepsilon \pounds_{mt}$ and C $\varepsilon \pounds_{tn}$. This decomposition is called the factor rank factorization and $\rho_f\,(A) = t$.

Example 2.2.4

Let us consider the incline $\pounds = \{0, a, b, c, d, 1\}$ in Example (2.1.8).

Let $A = \begin{pmatrix} d & d \\ d & d \end{pmatrix} \varepsilon \pounds_2$

$$A = \begin{pmatrix} 1 \\ d \end{pmatrix} \begin{pmatrix} b & c \end{pmatrix} = BC$$

Clearly, $A= BC$ is the factor rank factorization of A and $\rho_f\,(A) = 1$.

Remark 2.2.5

The row rank, column rank and the factor rank of a zero matrix is zero. For a finite matrix A, $\rho_r\,(A)$ is the maximum number linearly independent rows of A.

Definition 2.2.6

Let $A \varepsilon \pounds_{mn}$ with $\rho_r(A) = r$. Then there exist matrices $B \varepsilon \pounds_{mr}$ and $C \varepsilon \pounds_{rn}$ such that $\rho_r(A) = \rho_r(C)=r$ and $A =BC$. This decomposition is called a row rank factorization of A.

Definition 2.2.7

Let $A \varepsilon \pounds_{mn}$ with $\rho_c(A) = s$. Then there exists matrices $B \varepsilon \pounds_{ms}$ and $C \varepsilon \pounds_{sn}$ such that $\rho_c(A) = \rho_c(B) = s$ and $A=BC$. This decomposition is called a column rank factorization of A.

Definition 2.2.8

Let $A \varepsilon \pounds_{mn}$ with $\rho(A) = \rho_r(A) = \rho_c(A) = r$, then there exists matrices $B \varepsilon \pounds_{mr}$ and $C \varepsilon \pounds_{rn}$ such that $A = BC$ and $\rho(A) = \rho_c(B) = \rho_r(C) = r$. This is called a rank factorization of A.

Example 2.2.9

Let us consider the incline in Example (2.2.2).

Let $A = \begin{pmatrix} 1 & 0.8 & 0 \\ 0.8 & 0.7 & 0 \\ 0.7 & 0.6 & 0 \end{pmatrix}$

$(0.8, 0.7, 0.6)^T$ and $(1, 0.8, 0.7)^T$ are linearly independent. Hence $\rho_c(A) = 2$.

Each row of A is not a linear combination of the other two rows, that is, all three rows are linearly independent. Hence $\rho_r(A) = 3$.

$$A = \begin{pmatrix} 1 & 0.8 \\ 0.8 & 0.7 \\ 0.7 & 0.6 \end{pmatrix} \begin{pmatrix} 1 & 0.6 & 0 \\ 0.7 & 0.8 & 0 \end{pmatrix} = BC$$

Where, $B \varepsilon £_{32}$ and $C \varepsilon £_{23}$; $A = BC$ is the factor rank factorization. Hence $\rho_f(A) = 2$.

Therefore $\rho_f(A) = \rho_c(A) = 2$ and $\rho_r(A) = 3$. The decomposition $A = BC$ with $B \varepsilon £_{32}$, $C \varepsilon £_{23}$, is a column rank factorization and also the factor rank factorization. However, $A = BC$ is not a row rank factorization.

Remark 2.2.10

For field based matrices A,B if $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and ranks are equal then $\mathcal{R}(A) = \mathcal{R}(B)$. However, this fails for fuzzy matrices and thereby fails for matrices over an incline. This can be seen by the following example:

Example 2.2.11

Let us consider the incline $£ = \{0, a, b, c, d, 1\}$ in Example (2.1.8).

Let $A = \begin{pmatrix} d & d \\ d & d \end{pmatrix} \varepsilon £_2$ and $B = \begin{pmatrix} 1 & b \\ d & 0 \end{pmatrix} \varepsilon £_2$

$$A = \begin{pmatrix} b & d \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ d & 0 \end{pmatrix} = \begin{pmatrix} d & d \\ d & d \end{pmatrix} = XB$$

Hence by Lemma (1.2.8), $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.

Here, $(1,b) \neq \alpha\,(d,d) + \beta\,(d,d)$ for $\alpha, \beta \varepsilon £$.

Therefore, 1st row of B is not a linear combination of the rows of A. Thus $\mathcal{R}(B) \not\subseteq \mathcal{R}(A)$ and $\rho_r(A) = \rho_c(A) = 1$, $\rho_r(B) = \rho_c(B) = 2$ and $\rho_r(X) = \rho_c(X) = 2$. Thus $\rho_r(A) \neq \rho_r(B)$ and $\rho_c(A) \neq \rho_c(X)$. Hence $A = XB$ is neither a row rank factorization nor a column rank factorization.

Definition 2.2.12

For vectors $u \; \varepsilon \; £^m$ and $v \; \varepsilon \; £^n$ the cross vector $c(u,v)$ is the matrix $A = (a_{ij}) \; \varepsilon \; £_{mn}$ such that $a_{ij} = u_i v_j$, that is, $A = v^T u$.

Lemma 2.2.13

A non zero matrix A is a cross vector if and only if $\rho_r(A) = 1$ if and only if $\rho_c(A) = 1$ if and only if $\rho_f(A) = 1$.

Proof

If A is a cross vector, then by Definition (2.2.12), $A = v^T u$. Since A is non zero matrix, u and v are non zero vectors. Hence $\rho_r(u) = \rho_r(v) = 1$ and $\rho_c(v^T) = 1$. Therefore, $\rho_r(A) = \rho_c(A) = 1$.

Conversely, if $\rho_r(A) = 1$, then by Definition (2.2.6), A has a row rank factorization of the form $A=BC$ where B^T and C are row vectors. Hence A is a cross vector. Similarly when $\rho_c(A) = 1$. By Definition (2.2.12), A is a cross vector. Again if A is a cross vector, then $A = v^T u$ is a factor rank factorization of A and $\rho_f(A) = 1$.

Conversely if $\rho_f(A) = 1$, then A has a factor rank factorization of the form $A = BC$ with B^T and C as row vectors. Thus A is a cross vector.

Now we shall see the properties of the factor rank and its relation with row rank and column rank of incline matrix.

Proposition 2.2.14

Let $A \varepsilon \pounds_{mn}$. The factor rank $\rho_f(A)$ satisfy the following properties :

(i) $\rho_f(A) \leq min \{\rho_r(A), \rho_c(A)\}$

(ii) $\rho_f(PAQ) \leq \rho_f(A)$, if the matrix PAQ is defined

(iii) $\rho_f(A)$ is the smallest size of a set S of vectors such that $\mathcal{R}(A) \subseteq <S>$

(iv) $\rho_f(A)$ is the least number of rank 1 matrices whose sum is A.

Proof

(i) Let $\rho_r(A) = r$, then A has a row rank factorization $A=BC$ with $\rho_r(A) = \rho_r(C) = r$, $B \varepsilon \pounds_{mr}$ and $C \varepsilon \pounds_{rn}$. Then by Definition (2.2.3), $\rho_f(A) \leq \rho_r(A)$. Similarly, by Definition (2.2.7), $\rho_f(A) \leq \rho_c(A)$. Therefore, $\rho_f(A) \leq min \{\rho_r(A), \rho_c(A)\}$. Thus (i) holds.

(ii) Let $\rho_f(A) = t$, then by Definition (2.2.3), t is the least integer such that $A=BC$ with $B \varepsilon \pounds_{mt}$, $C \varepsilon \pounds_{tn}$ is the factor rank factorization of A. Then for $P \varepsilon \pounds_{pm}$, $Q \varepsilon \pounds_{nq}$, $PAQ = (PB)(CQ) = VW$, where $V \varepsilon \pounds_{pt}$ and $W \varepsilon \pounds_{tq}$, $Q \varepsilon \pounds_{nq}$. Thus we have decomposition for PAQ. Therefore by Definition (2.2.3), $\rho_f(PAQ) \leq t = \rho_f(A)$. Thus $\rho_f(PAQ) \leq \rho_f(A)$ and (ii) holds.

(iii) By Definition of factor rank, $A=BC$ is a decomposition with $B \varepsilon \pounds_{mt}$ and $C \varepsilon \pounds_{tn}$ where t is the smallest integer. By Lemma (1.2.9), $\mathcal{R}(A) = \mathcal{R}(BC) \subseteq \mathcal{R}(C) = <S>$ where S is the smallest spanning set of $\mathcal{R}(C)$. Thus (iii) holds.

(iv) Let s be the least number of rank 1 matrices A_i's such that $A = A_1+A_2 + \ldots\ldots +A_s$. Since each A_i is of rank 1, by Lemma (2.2.13), each A_i is a cross vector, that is, there exist vectors $v_i \varepsilon \pounds^m$

and $w_i \varepsilon \pounds^n$ such that $A_i = v_i^T w_i$ for each $i=1$ to s. Let $V \varepsilon \pounds_{mt}$ and $W \varepsilon \pounds_{tn}$ be defined such that the ith column of V is v_i^T and ith row of W is w_i. Then $A= VW$ is a factor rank decomposition, that is, $\rho_f(A) = s$. Thus (iv) holds.

Corollary 2.2.15

Let $A \varepsilon \pounds_{mn}$ and $B \varepsilon \pounds_{np}$. Then the following hold:

(i) $\rho_f (AB) \leq min \{\rho_r(A), \rho_r(B)\}$

(ii) $\rho_f(AB) \leq min \{\rho_c(A), \rho_c(B)\}$

(iii) $\rho_f(AB) \leq min \{\rho_r(A), \rho_r(B), \rho_c(A), \rho_c(B)\}$

Proof

(i) $\rho_f(AB) \leq \rho_f(A)$ (by Proposition (2.2.14 (ii))

$\leq \rho_r(A)$ (by Proposition (2.2.14 (i))

$\rho_f(AB) \leq \rho_f(B)$ (by Proposition (2.2.14 (ii))

$\leq \rho_r(B)$ (by Proposition (2.2.14 (i))

Hence $\rho_f (AB) \leq min \{\rho_r(A), \rho_r(B)\}$. Thus (i) holds.

(ii) Similarly (ii) can be proved that $\rho_f(AB)\leq min \{\rho_c(A), \rho_c(B)\}$ and (ii) holds.

(iii) This is directly follows from (i) and (ii).

Corollary 2.2.16

For $A \varepsilon \pounds_{mn}$, $B \varepsilon \pounds_{np}$,

(i) if $\rho_r(AB) = \rho_f(AB)$ then $\rho_r(AB) \leq min \{\rho_r(A), \rho_r(B)\}$

(ii) if $\rho_c(AB) = \rho_f(AB)$ then $\rho_c(AB) \leq min \{\rho_c(A), \rho_c(B)\}$

Proof

This follows from Corollary (2.2.15).

Corollary 2.2.17

For $A \varepsilon £_{mn}$,

$\rho_f(AA^T) \leq min\ \{\rho_r(A), \rho_c(A)\}$

$\rho_f(A^TA) \leq min\ \{\rho_r(A), \rho_c(A)\}$

Proof

This follows from Corollary (2.2.16), by replacing B by A^T and using $\rho_r(A^T) = \rho_c(A)$.

Definition 2.2.18

Let $A\ \varepsilon\ £_{mn}$. If $A= BC$ such that $\mathcal{R}(A) = \mathcal{R}(C)$ $(\mathcal{C}(A) = \mathcal{C}(B))$ then it is said to be a full row (column) space factorization of A. Further, if $A = BC$ with $\mathcal{R}(A) = \mathcal{R}(C)$ and $\mathcal{C}(A) = \mathcal{C}(B)$ then it is said to be a full space factorization.

Example 2.2.19

Let us consider the incline $£ = \{0,a,b,c,d,1\}$ in Example (2.1.8).

Let $A= \begin{pmatrix} d & d \\ d & d \end{pmatrix}\ \varepsilon\ £_2$, $B =\begin{pmatrix} 1 & 0 \\ d & 0 \end{pmatrix}\ \varepsilon\ £_2$

and $C= \begin{pmatrix} b & c \\ 0 & 0 \end{pmatrix} \varepsilon\ £_2$.

Then $A = BC$, by Lemma (1.2.8), $\mathcal{R}(A) \subseteq \mathcal{R}(C)$ and $\mathcal{C}(A) \subseteq \mathcal{C}(B)$. Here, $C =\begin{pmatrix} b & c \\ 0 & 0 \end{pmatrix} \neq YA$ for all $Y \varepsilon £_2$ and $B = \begin{pmatrix} 1 & 0 \\ d & 0 \end{pmatrix} \neq AX$ for all $X \varepsilon £_2$.

Therefore, by Lemma (1.2.8), $\mathcal{R}(C) \nsubseteq \mathcal{C}(A)$ and $\mathcal{C}(B) \nsubseteq \mathcal{C}(A)$. Thus $\mathcal{R}(A) \neq \mathcal{R}(C)$ and $\mathcal{C}(A) \neq \mathcal{C}(B)$. Hence A=$BC$ *is* neither a row space factorization nor a column space factorization.

2.3. Zero Patterns of Incline Matrices

In this section, we shall discuss the relation between the greatest element '1' of an incline and the multiplicative identity "$1_£$" of an incline £ and then we derive some basic properties of zero patterns of a matrix over an incline.

Definition 2.3.1

Let $x,y \varepsilon £$, if $x \leq y$ for all $y \varepsilon £$ then x is called the least element and denoted as '0'. If $x \geq y$ for all $y \varepsilon £$ then x is called the greatest element and denoted as '1'.

Proposition 2.3.2

In an incline £, Zero element of £ coincides with least element of £.

Proof

0 is the least element of £

$\Leftrightarrow 0 \leq x$ for all $x \varepsilon £$ (by Definition (2.3.1))

$\Leftrightarrow 0x = x0=0$ for all $x \varepsilon £$ and

$0 + x = x+0 = x$ (by Incline Property (1.2.2) and by Definition (2.3.1))

$\Leftrightarrow 0$ is the zero element of £.

Proposition 2.3.3

If an incline £ has the multiplicative identity "$1_£$" then it coincides with its greatest element '1'.

Proof

By Definition (1.2.15) of $1_£$, $1_£x = x1_£ = x$, for all $x \varepsilon £$. $\rightarrow$ (2.3.1)

By Definition (2.3.1), $x \leq 1$, for all $x \varepsilon £$. $\rightarrow$ (2.3.2)

From (2.3.1), (2.3.2) and by Incline Property (1.2.2), we have

$1 = 1_£ 1 \leq 1_£ \leq 1 \Rightarrow 1 = 1_£$.

Example 2.3.4

Consider the incline $£ = \{[0,1], +, \cdot\}$, where + is the maximum and $\cdot$ is the usual multiplication. In this incline, 1 is the multiplicative identity as well as the greatest element, that is, $1_£ = 1$.

Remark 2.3.5

In an linearly ordered incline, if the multiplication operation is the minimum, that is, $xy = min\ \{x,y\}$ then the greatest element '1' reduces to "$1_£$", for $x\ \varepsilon\ £,\ x \leq 1$ implies $x1 = x = 1x$ for all $x\ \varepsilon\ £$ which implies $1 = 1_£$.

Remark 2.3.6

The converse of Proposition (2.3.3) need not be true. This can be seen from the following example:

Example 2.3.7

In Example (2.1.8), 1 is the greatest element but it is not the multiplicative identity, since, $1x \neq x$, for all $x\ \varepsilon\ £$.

Lemma 2.3.8

In the incline $£$, the greatest element '1' is the multiplicative identity for elements of DL. In particular, for a regular incline $£$, '1' coincides with the multiplicative identity "$1_£$" of $£$.

Proof

Let a be an arbitrary element of DL, then a is idempotent. By Incline Property (1.2.2)

$$a = a^2 = aa \leq a1 \leq a \Rightarrow a{=}a1.$$

Similarly, $a = a^2 = aa \leq 1a \leq a \Rightarrow a{=}1a$.

Thus, $a1 = 1a{=}a$ for all $a\ \varepsilon\ DL$. 1 is the multiplicative identity for the elements of DL.

For a regular incline, by Proposition (1.2.20) $DL{=}\ £$, 1 is the multiplicative identity of $£$.

Remark 2.3.9

We observe that, in general, 1 is the multiplicative identity of DL need not imply $DL = £$ and $1 \notin DL$. This can be seen by the following example:

Example 2.3.10

For the incline £ in Example (2.1.8), 0 is the least element and 1 is the greatest element of $£$, $DL = \{0, d\}$. Therefore, £ is not a regular incline. 1 is the multiplicative identity for DL and $1 \notin DL$. Here, £ has no multiplicative identity. For instance $a \varepsilon £$, $a1 \neq a$.

Lemma 2.3.11

In an incline £, whose elements are all linearly ordered then for $\alpha,\beta,\gamma \ \varepsilon £$

(i) if $\alpha + \beta \geq \gamma$ then either $\alpha \geq \gamma$ *(or)* $\beta \geq \gamma$

(ii) if $\alpha\beta \geq \gamma$ then either $\alpha \geq \gamma$ and $\beta \geq \gamma$.

Proof

(i) Let $\alpha + \beta \geq \gamma$ if $\alpha \leq \gamma$ and $\beta \leq \gamma$ then $\alpha + \beta \leq \gamma + \gamma = \gamma$. Which is a contradiction. Therefore either $\alpha \geq \gamma$ (or) $\beta \geq \gamma$. Thus (i) holds.

(ii) Let $\alpha\beta \geq \gamma$, if $\alpha \leq \gamma$ or $\beta \leq \gamma$, then $\alpha\beta \leq \alpha \leq \gamma$ (or) $\alpha\beta \leq \beta \leq \gamma$. Which is a contradiction. Hence $\alpha \geq \gamma$ and $\beta \geq \gamma$. Thus (ii) holds.

Proposition 2.3.12

If £ be a regular incline whose elements are all linearly ordered then £ is commutative.

Proof

For x, $y \ \varepsilon £$, either $x \leq y$ (or) $y \leq x$. If $x \leq y$ then by Proposition (1.2.23), $y \varepsilon \ x \ \{1\}$ and by Lemma (1.2.22), $xy = yx = x$ (or) if $y \leq x$, then in the same manner, we get $yx = xy = y$. Thus in either case, £ is commutative.

Lemma 2.3.13

Let $£$ be a regular incline whose elements are all linearly ordered. For $\alpha, \beta \,\varepsilon\, £$, $\alpha \leq \beta \Leftrightarrow \alpha + \beta = \beta \Leftrightarrow \alpha\beta = \alpha$.

Proof

For $\alpha,\beta \,\varepsilon\, £$, by incline order relation, $\alpha \leq \beta \Leftrightarrow \alpha + \beta = \beta$. By multiplying α, on both sides yields, $\alpha^2 + \alpha\beta = \alpha\beta$ which implies by Proposition (1.2.19) that $\alpha + \alpha\,\beta = \alpha\beta$, again by incline order relation it follows that $\alpha \leq \alpha\beta$. By incline Property (1.2.2), $\alpha\beta \leq \alpha$, hence we get $\alpha\beta = \alpha$.

Henceforth, let us consider the incline with zero element denoted as *'0'* and greatest element denoted as *'1'* and which has no multiplicative identity. For, if it has a multiplicative identity "$1_£$" then it coincides with the greatest element by Proposition (2.3.3). The greatest element *'1'* is the multiplicative identity of *DL* and by Lemma (2.3.8.) for a regular incline $1 = 1_£$, that is, *'1'* coincides with the multiplicative identity "$1_£$".

Definition 2.3.14

Let $£$ be an incline whose elements are all linearly ordered with least element 0 and greatest element 1. For $A = (a_{ij}) \,\varepsilon\, £_{mn}$ and $\alpha \,\varepsilon\, £$, the zero pattern A_α of A is defined as

$$(A_\alpha)_{ij} = \begin{cases} 1 & \text{if } a_{ij} \geq \alpha \\ 0 & \text{otherwise} \end{cases}$$

Let ϕ_A denotes the set of all non zero entries of A then for each $\alpha \,\varepsilon\, \phi_A$, the zero patterns A_α of A is the matrix, whose entries are 0 and 1.

Remark 2.3.15

The elements of $£$ to be linearly ordered in $£$ is essential. This is illustrated in the following example:

Example 2.3.16

Let us consider the incline $£ = (\mathcal{P}(D), \cup, \cap)$, $D = \{a,b,c\}$, where $\mathcal{P}(D)$ is the power set of D and set inclusion as the order relation "$\leq$".

Here, the incline $£$ is regular whose elements are all idempotent but elements are all not linearly ordered. For instance, the elements $\{a\}$, $\{b\}$ and $\{c\}$ are not comparable. Hence the zero patterns are not defined for all matrices over $£$.

For instance, $A = \begin{pmatrix} \{b\} & \{c\} \\ \{c\} & \{a\} \end{pmatrix}$, has no zero pattern.

Proposition 2.3.17

Let $£$ be an incline whose elements are all linearly ordered. Let $A \, \varepsilon \, £_{mn}$. ϕ_A be the set of all non zero entries of A. Then for $\alpha \geq \beta \, \varepsilon \, \phi_A$, the zero patterns $A_\alpha \leq A_\beta$.

Proof

Let $A_\alpha = (A_\alpha)_{ij}$ and $A_\beta = (A_\beta)_{ij}$ if $(A_\alpha)_{ij} = 0$ then $(A_\alpha)_{ij} \leq (A_\beta)_{ij}$. If $(A_\alpha)_{ij} = 1$ then by Definition (2.3.14), we have $a_{ij} \geq \alpha \geq \beta \Rightarrow (A_\beta)_{ij} = 1$. Thus $(A_\alpha)_{ij} \leq (A_\beta)_{ij}$ for all i and j. Hence $A_\alpha \leq A_\beta$ for all $\alpha \geq \beta$.

Theorem 2.3.18

Let $£$ be an incline whose elements are all linearly ordered. Let $A \, \varepsilon \, DL_{mn}$ and ϕ_A be the set of all non zero entries of A. Then $A = \sum_{\alpha \varepsilon \phi_A} \alpha A_\alpha$.

Proof

Let $A = (a_{ij}) \, \varepsilon \, DL_{mn}$ and $B = \sum_{\alpha \varepsilon \phi_A} \alpha A_\alpha$. We claim that $B = (b_{ij}) \, \varepsilon \, DL_{mn}$. By Definition (2.3.14) of $A_\alpha, (A_\alpha)_{ij} = 1$ (or) 0; according as $a_{ij} \geq \alpha$ (or) $a_{ij} < \alpha$. Since $A \, \varepsilon \, DL_{mn}$ and by Lemma (2.3.8), 1 is the multiplicative identity of DL. Hence $1\alpha = \alpha$, for $\alpha \, \varepsilon \phi_A$. Thus $B \, \varepsilon \, DL_{mn}$.

If $B = (b_{ij})\ \varepsilon\ DL_{mn}$ then it is enough to show that $a_{ij} = b_{ij}$ for all i,j. Since $a_{ij}\ \varepsilon\ \phi_A$, a_{ij} is the (ij)th entry of $\alpha\, A_\alpha$ for some α, one of the summands of B.

On the other hand, $0 \neq b_{ij} = \left(\sum_{\alpha\varepsilon\phi_A} \alpha A_\alpha\right)_{ij} = \sum_{\alpha\varepsilon\phi_A} (ij)$th entry of (αA_α)

$= \sum_{\alpha\varepsilon\phi_A} \alpha (ij)$th entry of A_α

$= \alpha$(or)0 for $\alpha\varepsilon\phi_A$

$= a_{ij}$, for all i,j

We have $b_{ij} = a_{ij}$, for all i, j.

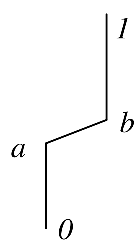
Hence $A = \sum_{\alpha\varepsilon\phi_A} \alpha A_\alpha$.

Remark 2.3.19

In Theorem (2.3.18), the condition $A\ \varepsilon\ DL_{mn}$ is essential. This is illustrated in following example:

Example 2.3.20

Consider the incline $£ = \{0, a, b, 1\}$, lattice ordered by the following Hasse diagram.



Define $£ \times £ \rightarrow £$ as follows

$$xy = \begin{cases} a \text{ if } x, y\ \varepsilon\ \{a,b,1\} \\ 0 \text{ otherwise} \end{cases}$$

Where the addition $(+)$ and multiplication $(\cdot)$ is defined as in the following table:

+	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

·	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	a
1	0	a	a	a

In this finite incline the elements are all linearly ordered and $DL = \{0, a\}$.

Let $A = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} \varepsilon \pounds_{mn}$ and $\phi_A = \{a,b,1\}$ be the non zero entries of A.

$$A_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad aA_a = \begin{pmatrix} a & a \\ 0 & a \end{pmatrix}$$

$$A_b = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad bA_b = \begin{pmatrix} 0 & a \\ 0 & a \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad 1A_1 = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$$

$\sum_{\alpha\varepsilon\phi_A} \alpha A_\alpha = \begin{pmatrix} a & a \\ 0 & a \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} = $ A. Here, $B = \sum_{\alpha\varepsilon\phi_A} \alpha A_\alpha$ and A are not comparable.

Thus Theorem (2.3.18) fails.

Corollary 2.3.21

Let $\pounds$ be a regular incline whose elements are all linearly ordered and $A \ \varepsilon \pounds_{mn}$, ϕ_A be the set of all non - zero entries of A. Then $A = \sum_{\alpha\varepsilon\phi_A} \alpha A_\alpha$.

Proof

Since £ is regular incline by Proposition (1.2.20), each element of £ is idempotent and therefore DL = £. Then the rest follows from Theorem (2.3.18), under the condition that all elements are linearly ordered.

Remark 2.3.22

The expression $A = \sum_{\alpha\varepsilon\phi_A} \alpha A_\alpha$ is called the resolution of A. This can be illustrated by the following example:

Example 2.3.23

Let us consider the incline $\pounds = \{[0,1], sup(x,y), min\ (x,y)\}$. $\pounds$ is a regular incline, since the elements are all idempotent. In this incline, the elements are all linearly ordered.

Let $A = \begin{pmatrix} 0.5 & 0.3 \\ 0.4 & 1 \end{pmatrix} \varepsilon \pounds_2$

and $\phi_A = \{0.3, 0.4, 0.5, 1\}$ is the set of non zero entries of A.

$A_{0.3} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \varepsilon\, £_2$ $\quad A_{0.4} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \varepsilon\, £_2$

$A_{0.5} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \varepsilon\, £_2$ $\quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \varepsilon\, £_2$

$0.3A_{0.3} = \begin{pmatrix} 0.3 & 0.3 \\ 0.3 & 0.3 \end{pmatrix}$ $\quad 0.4A_{0.4} = \begin{pmatrix} 0.4 & 0 \\ 0.4 & 0.4 \end{pmatrix}$

$0.5A_{0.5} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$ $\quad 1A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Hence $A = 0.3A_{0.3} + 0.4A_{0.4} + 0.5A_{0.5} + 1A_1$.

Remark 2.3.24

Since by Proposition (1.2.21), a commutative regular incline is a distributive lattice and $DL = £$, Theorem (2.3.18) reduces to the following:

Corollary 2.3.25

Let $£$ be a distributive lattice whose elements are all linearly ordered and $A\ \varepsilon\, £_{mn}$, ϕ_A be the set of all non - zero entries of A. Then $A = \sum_{\alpha \varepsilon \phi_A} \alpha A_\alpha$.

Remark 2.3.26

By Lemma (2.3.13), the incline operations reduces to the max-min composition in $£$. Therefore Corollary (2.3.21), reduces to the result on zero patterns of fuzzy matrix due to Kim and Roush [28].

Corollary 2.3.27

Let $A\ \varepsilon\ \mathcal{F}_{mn}$ and ϕ_A be the set of all non - zero entries of A. Then $A = \overset{\oplus}{\underset{\alpha \varepsilon \phi_A}{}} \alpha A_\alpha$; that is , A is expressed as a fuzzy linear combination of its zero patterns.

Remark 2.3.28

For fuzzy matrices as in Boolean matrices, it is well known that A is idempotent if and only if each zero pattern of *A* is idempotent, result due to Kim and Roush [28], (p.43, [36]). However this fails for matrices over an incline.

Example 2.3.29

Let us consider the incline *£* = *{[0,1], sup (x,y), (xy)}*, where the addition as maximum and usual multiplication.

Let $A = \begin{pmatrix} 0.5 & 0.3 \\ 0.4 & 0.7 \end{pmatrix} \varepsilon\, £_2$

Here, $A^2 \neq A$, *A* is not an idempotent matrix.

For this *A*, $\phi_A = \{0.3, 0.4, 0.5, 0.7\}$

The zero patterns are

$A_{0.3} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \varepsilon\, £_2$ $\quad$ $A_{0.4} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \varepsilon\, £_2$

$A_{0.5} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \varepsilon\, £_2$ $\quad$ $A_{0.7} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \varepsilon\, £_2$

Each zero pattern of *A* is idempotent but *A* is not idempotent.

Theorem 2.3.30

Let *£* be an incline whose idempotent elements are linearly ordered. $A\ \varepsilon\ DL_n$ is idempotent if and only if each zero pattern of *A* is idempotent.

Proof

Let us denote the distinct entries of *A* by $\{a_1, a_2, \ldots, a_k\}$ and A_i for $i=1$ to k be the corresponding zero patterns. With this notation, the resolution *A* is given by

$$A = \sum_{i=1}^{k} a_i A_i, \text{ where each } a_i \text{ is idempotent}$$

Then
$$A^2 = \left(\sum_{i=1}^{k} a_i A_i\right)\left(\sum_{i=1}^{k} a_j A_j\right)$$
$$= \sum_{i=1}^{k} a_i A_i^2 + \sum_{\substack{i \neq j \\ i=1}}^{k} a_i a_j A_i A_j + \sum_{\substack{i \neq j \\ i=1}}^{k} a_j a_i A_j A_i$$

Suppose A_i's are idempotent, then

$A^2 = (\sum_{i=1}^{k} a_i A_i) + \sum_{\substack{i \neq j \\ i=1}}^{k} a_i a_j A_i A_j + \sum_{\substack{i \neq j \\ i=1}}^{k} a_j a_i A_j A_i$

By Proposition (2.3.17), A_i's are comparable, that is, for $a_i > a_j$, $A_i \leq A_j$. Hence $A_i A_j \leq A_j^2 = A_j$ and $A_j A_i \leq A_j^2 = A_j$. $a_i a_j \leq a_i$ and $a_{ij} \leq a_j$. Therefore for $a_i \geq a_j$ we have $a_i a_j A_i A_j \leq a_j A_j$ and $a_j a_i A_j A_i \leq a_j A_j$. By Proposition (1.2.14), $a_i a_j A_i A_j + a_j A_j = a_j A_j$ and $a_i a_j A_j A_i + a_j A_j = a_j A_j$. Adding up all these k equations $A^2 = \sum_{i=1}^{k} a_i A_i + \sum_{i=1}^{k} a_j A_j = A$.

Thus A is idempotent. Converse can be proved in the same manner.

Theorem 2.3.31

Let A ε $£_n$ be an idempotent matrix. Let k be an integer with $1 \leq k \leq n$. Let $\tilde{A} = (\tilde{a}_{ij})$ be the matrix obtained from $A = (a_{ij})$ by replacing the kth row A_{k*} of A with $a_{kk} A_{k*}$. Then $\tilde{A}$ is also an idempotent matrix, $\rho_r(A) = \rho_r(\tilde{A})$ and $\rho_f(A) = \rho_f(\tilde{A})$.

Proof

Since A is idempotent, $A^2 = A$, by Definition (1.2.28) of incline composition.

$$A_{k*} = a_{kk} A_{k*} + \sum_{j \neq k} a_{kj} A_{j*}$$

$$= \tilde{A}_{k*} + \sum_{j \neq k} a_{kj} \tilde{A}_{j*} \qquad \rightarrow (2.2.4)$$

$$\tilde{A}_{k*} = \text{kth row of } \tilde{A}$$

$$= a_{kk} A_{k*}$$

$$= a_{kk} (\tilde{A}_{k*} + \sum_{j \neq k} a_{kj} \tilde{A}_{j*})$$

$$\tilde{A}_{k*} = \tilde{a}_{kk} \tilde{A}_{k*} + \sum_{j \neq k} a_{kj} \tilde{A}_{j*} \qquad \rightarrow (2.2.5)$$

Since $\tilde{a}_{kj} = a_{kk} a_{kj}$ and $\tilde{A}_{j*} = A_{j*}$ for $j \neq k$.

For $i \neq k$, by (2.2.4)

$$\tilde{A}_{i*} = A_{i*} = a_{ik} A_{k*} + \sum_{j \neq k} a_{ij} A_{j*}$$

$$= a_{ik} (\tilde{A}_{k*} + \sum_{j \neq k} a_{kj} \tilde{A}_{j*}) + \sum_{j \neq k} \tilde{a}_{ij} \tilde{A}_{j*}$$

$= \quad \tilde{a}_{ik}\,\tilde{A}_{k*} + \sum_{j\neq k} \tilde{a}_{ij}\,\tilde{A}_{j*}$ $\rightarrow$(2.2.6)

$a_{ij} \quad = \quad \tilde{a}_{ij}$ for $i\neq k$

(ij) th entry of A = *(ij)*th entry of A^2

$a_{ij} \;=\; \sum_{k=1}^{n} a_{ik}\, a_{kj} \geq a_{ik}\, a_{kj}$ (by Incline Property 1.2.1)

$a_{ij} \quad \geq a_{ik}\, a_{kj}$

Since $\tilde{a}_{ij}\,\tilde{A}_{j*} \geq a_{ik}\, a_{kj}\tilde{A}_{j*}$, when $i \neq k$, we have $(\tilde{A})^2 = \tilde{A}$. Therefore, $\tilde{A}$ is idempotent.

Further by (2.2.4) and (2.2.6), $\mathcal{R}(A) \subseteq \mathcal{R}(\tilde{A})$ and by (2.2.5) and (2.2.6), $\mathcal{R}(\tilde{A}) \subseteq \mathcal{R}(A)$. Therefore, $\mathcal{R}(A) = \mathcal{R}(\tilde{A})$ and $\rho_r(A) = \rho_r(\tilde{A})$.

Since $\mathcal{R}(A) = \mathcal{R}(\tilde{A})$, by Lemma (1.2.8), $A = X\tilde{A}$ for some $X \,\varepsilon\, £_n$. By Proposition (2.2.14), $\rho_f(A) \leq \rho_f(\tilde{A})$. Again from $\mathcal{R}(\tilde{A}) \subseteq \mathcal{R}(A)$, by Lemma (1.2.8), $\tilde{A} = YA$ for some $Y \,\varepsilon\, £_m$. By Proposition (2.2.14), $\rho_f(\tilde{A}) \leq \rho_f(A)$. Therefore $\rho_f(A) = \rho_f(\tilde{A})$.

Hence the Theorem.

Theorem 2.3.32

Let $£$ be an incline whose elements are all linearly ordered. Let $A,B \,\varepsilon\, DL_n$ and $\alpha \,\varepsilon\, £$ then $(AB)_\alpha = A_\alpha\, B_\alpha$.

Proof

Let $A = (a_{ij}) \,\varepsilon\, £_n$ and $B = (b_{ij}) \,\varepsilon\, £_n$

$((AB)_\alpha)_{ij} = 1 \Rightarrow (AB)_{ij} \geq \alpha$ (by Definition (2.3.14))

⇨ $\sum_{k=1}^{n} a_{ik}b_{kj} \geq \alpha$

⇨ $a_{ik}b_{kj} \geq \alpha$ for at least one k (by Lemma (2.3.11(i)))

⇨ $a_{ir}\, b_{rj} \geq \alpha$ (say $k= r$).

Hence $a_{ir}\, b_{rj} \geq \alpha \Rightarrow a_{ir} \geq \alpha$ and $b_{rj} \geq \alpha$ for some r by Lemma (2.3.11(ii)).
By Definition (2.3.14), $(A_\alpha)_{ir} = 1$ *and* $(B_\alpha)_{rj}=1$

⇨ $\sum_{k=1}^{n} (A_\alpha)_{ik}\, (B_\alpha)_{kj} = 1$, since $A,B \,\varepsilon\, DL_n$, by Lemma (2.3.8), *1* is the multiplicative identity of *DL*.

$\Rightarrow$ $(A_\alpha B_\alpha)_{ij} = 1$.

Similarly, if $((AB)_\alpha)_{ij} = 0$ then $(A_\alpha B_\alpha)_{ij} = 0$ can be proved.

Thus $(AB)_\alpha = A_\alpha B_\alpha$.

Remark 2.3.33

In Theorem (2.3.32), the condition $A,B \ \varepsilon \ DL_n$ is essential. This is illustrated in the following example:

Example 2.3.34

Consider the incline $£ = \{0,a,b,1\}$ in Example (2.3.20).

Let $A = \begin{pmatrix} a & b \\ 1 & a \end{pmatrix} \varepsilon £_2$ and $B = \begin{pmatrix} a & 0 \\ a & a \end{pmatrix} \varepsilon DL_2$

For A, $\phi_A = \{a,b,1\}$ and for B, $\phi_B = \{a\}$ are the non - zero entries of A and B.

The Zero patterns of A,

$A_a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $A_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

The Zero patterns of B,

$$B_a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We have, $AB = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$

The Zero patterns of AB,

$$AB_a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Therefore, $AB_\alpha \neq A_\alpha B_\alpha$. Thus Theorem (2.3.32) fails.

In particular, for a regular incline Theorem (2.3.32) reduces to the following:

Corollary 2.3.35

Let $£$ be a regular incline whose elements are all linearly ordered. If $A,B \ \varepsilon \ £_n$ and $\alpha \ \varepsilon \ £$ then $(AB)_\alpha = A_\alpha B_\alpha$.

2.4. Invertible Incline Matrices

In this section, we shall discuss the invertibility for matrices over *DL*, the set of idempotent elements. Let $R_n(£), G_n(£)$ and $P_n(£)$ denotes the set of all regular matrices, the set of all invertible matrices and the set of all permutation matrices of order n over £ respectively.

Definition 2.4.1

A matrix $P \ \varepsilon \ £_n$ is called a generalized permutation matrix if it has exactly one entry equals the greatest element '*1*', in each row and each column of *P* and the remaining entries are the least element '*0*'.

Remark 2.4.2

Since by Proposition (2.3.3), for any incline with multiplicative identity "$1_£$", it coincides with the greatest element '*1*'. Hence the permutation matrix $\Rightarrow$ generalized permutation matrix. Thus the Definition (2.4.1) coincides with the Definition (1.2.30).

Remark 2.4.3

We observe that, Generalized permutation matrix $\nRightarrow$ permutation matrix, this can be seen from the following example:

Example 2.4.4

In Example (2.1.8), £ has no multiplicative identity but it has the greatest element '*1*'. $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a generalized permutation matrix. By Definition (1.2.30), *P* is not a permutation matrix, since, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$.

Definition 2.4.5

Let £ be an incline with 0 and 1. $A\ \varepsilon\, DL_n$ is invertible if and only if there exists $X\ \varepsilon\, DL_n$ such that $AX= XA= I_n$, the diagonal matrix whose entries are all the greatest element '1'.

Remark 2.4.6

In particular for a regular incline '1' coincides with the multiplicative identity "$1_£$", and hence $I_n = (I_£)_n$. The above Definition (2.4.5) reduces to the Definition (1.2.29), invertible over £.

Example 2.4.7

In Example (2.1.8), 1is not the multiplicative identity for £ and $£_n$ has no permutation matrix.

For instance, $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ \varepsilon\ £_2$ and $P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ \varepsilon\ £_2$

are not permutation matrices in $£_2$.

$P_1P_1^T = P_1^TP_1 = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \neq I_2 \neq (I_£)_2$ and $P_2P_2^T = P_2^TP_2 = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \neq I_2 \neq (I_£)_2$

Here, P is not a invertible matrix.

Remark 2.4.8

The existence of the multiplicative identity "$1_£$", in an incline has no relation with the regularity of that incline. This can be seen from the following example:

Example 2.4.9 (a)

Consider the incline $£ = \{[0,1],\ +,\ \cdot\}$, where the addition is maximum and usual multiplication. Here, '1' is the multiplicative identity as well as the greatest element. But, £ is not regular.

Example 2.4.9 (b)

Consider the incline $£ = \{[0,1],\ sup(x,y),\ \ min\ (x,y)\}$. In this incline 1 is the multiplicative identity as well as the greatest element, that is, $1_£ = 1$ and also £ is regular.

Theorem 2.4.10

Let $£$ be an incline with 0 and 1. Then $A \varepsilon DL_n$ is invertible $\Leftrightarrow AA^T = A^TA = I_n,$ where I_n is the matrix whose diagonal entries are the greatest element '1' and the remaining elements are all zeros.

Proof

Since $A \varepsilon DL_n$ is invertible by Definition (1.2.29), $AX = XA = I$, for some $X \varepsilon DL_n$. Let $A = (a_{ij})$ and $X = (x_{ij})$. By $AX = I$, for any s,i,j $(i\neq j)$, it is easy to see that $a_{is}\ x_{sj} \leq \sum_{k=1}^{n} a_{ik}\ x_{kj} = 0$, that is, $a_{is}\ x_{sj} = 0$. By $XA=I$, for any $i(1\leq i\leq n)$, $\sum_{k=1}^{n} a_{ki} \geq \sum_{k=1}^{n} x_{ki}a_{ki}=1$ and $\sum_{k=1}^{n} x_{ik} \geq \sum_{k=1}^{n} x_{ik}a_{ki}=1$, that is, $\sum_{k=1}^{n} a_{ki} = \sum_{k=1}^{n} x_{ik}=1$, since a_{ij} and x_{ji} are idempotent.

Now, by using Lemma (2.3.8), we get

$$a_{ij} = a_{ij}\ 1 = a_{ij}\ (\textstyle\sum_{k=1}^{n} x_{jk}) = \sum_{k=1}^{n} a_{ij}\ x_{jk} = a_{ij}\ x_{jk} + \sum_{k=1}^{n} a_{ij}\ x_{jk} = a_{ij}\ x_{ij} + 0 = a_{ij}\ x_{ji}$$

$$= 0 + a_{ij}\ x_{ji} = \sum_{\substack{k=1\\ k\neq j}}^{n} a_{kj}x_{ji} + a_{ij}\ x_{ji} = \sum_{k=1}^{n} a_{kj}\ x_{ji} = (\textstyle\sum_{k=1}^{n} a_{kj})\ x_{ji} = 1\ x_{ji} = x_{ji}.$$

Hence $X=A^T$ and $AA^T = A^TA = I_n$. Converse is trivial.

Corollary 2.4.11

Let $£$ be a regular incline. Then $A \varepsilon £_n$ is invertible $\Leftrightarrow AA^T = A^TA = I_n$.

Proof

From Lemma (2.3.8), the greatest element '1', is the multiplicative identity for a regular incline and by Proposition (1.2.20), $DL = £$. Then the result follows from Theorem (2.4.10).

Remark 2.4.12

Since for a regular incline $£$ whose elements are all linearly ordered the incline operations reduces to max-min composition and by Lemma (2.3.13), $1_£ = 1$, the Theorem (2.4.10) reduces to the result of Zhao [49] (p.32, [36]).

Corollary 2.4.13

A be a matrix over the max - min Fuzzy algebra, then A is invertible $\Leftrightarrow$ A is a permutation matrix that is, $G_n(£) = P_n\,(£)$.

Remark 2.4.14

We observe that if $A\ \varepsilon\ £_n$ is invertible matrix over a regular incline then it need not be a permutation matrix. This is illustrated in the following:

Example 2.4.15

Let us consider the incline $£ = (\mathcal{P}(D), \cup, \cap)$ as in Example (2.3.16).

$$\text{Let} \quad A = \begin{pmatrix} \{a\} & \{b\} & \{c\} \\ \{b\} & \{c\} & \{a\} \\ \{c\} & \{a\} & \{b\} \end{pmatrix} \varepsilon\ £_3$$

Here, $£$ is a regular incline whose elements are all idempotent but the elements of $£$ are not linearly ordered.

$$\text{Then } AA^T = A^TA = \begin{pmatrix} D & \phi & \phi \\ \phi & D & \phi \\ \phi & \phi & D \end{pmatrix} = I_3\ \varepsilon\ £_3$$

A is invertible but A is not a permutation matrix.

Definition 2.4.16

A pair of non zero elements $x,y\ \varepsilon\ £$ is called an integral pair if $xy = 0$ and $x+y = 1$, where 0 and 1 are the least and greatest element of $£$ respectively.

Definition 2.4.17

An incline $£$ with 0 and 1 is called an integral incline if there do not exist an integral pair, where 0 and 1 are the least and greatest element of $£$ respectively.

Lemma 2.4.18

If idempotent elements $a,b\ \varepsilon\ £$ are comparable then (a,b) is not an integral pair.

Proof

Since $a,b \varepsilon £$ are comparable idempotent elements, either $a \leq b$ (or) $b \leq a$. Hence $a+b = b$ (or) $a+b = a$. $\rightarrow$ (2.4.1)

we claim that (a,b) is not an integral pair, for if a and b forms an integral pair then $a \neq 0$ and $b \neq 0$ such that $a+b = 1$ and $ab = 0$. Hence by (2.4.1) it follows that either $b=1$ (or) $a=1$, since a and b are idempotents by Lemma (2.3.8), $a1 = a$ and $1b = b$. Substitute $b = 1$ (or) $a = 1$ in $ab = 0$, then we get $0=ab = a1 = a \Rightarrow a=0$ (or) $0=ab = 1b=b \Rightarrow b=0$. Which is a contradiction. Hence the Lemma.

Theorem 2.4.19

Let £ be an incline whose idempotent elements are all linearly ordered. Then DL is an integral incline.

Proof

Since the idempotent elements are all linearly ordered, by Lemma (2.4.18), *DL* has no integral pair. Hence *DL* is an integral incline.

Remark 2.4.20

We observe that £ need not be an integral incline under the condition that the idempotent elements are linearly ordered. This is illustrated in the following example:

Example 2.4.21

Let us consider the incline $£ = \{0, a, b, c, d, 1\}$ in Example (2.1.8). Consider $a,d \varepsilon £$. Then $ad=0$ and $a+d = 1$. By Definition (2.4.17), it forms an integral pair. Hence £ is not an integral incline. However, the subincline $DL = \{0,d\}$ is an integral incline as it has no integral pair.

Remark 2.4.22

For a regular incline, since '*1*' coincides with the multiplicative identity "$1_£$" and whose elements are all linearly ordered then the Definition (2.4.17) coincides with the Definition (1.2.16). Hence Theorem (2.4.19) reduces to the following :

Corollary 2.4.23

Let *£* be a regular incline whose elements are all linearly ordered. Then *£* is an integral incline.

Remark 2.4.24

In the above Corollary (2.4.23), the converse need not be true. This is illustrated in the following example:

Example 2.4.25

Let us consider the incline $£ = \{[0,1], +, \cdot\}$, where "$\cdot$" denotes usual multiplication and the elements of *£* are all linearly ordered. For any pair of non zero elements x and y such that $xy \neq 0$. Hence, there does not exists integral pair and *£* is an integral incline and elements of *£* are linearly ordered but it is not regular.

Remark 2.4.26

Since by Proposition (1.2.21) and Proposition (1.2.20), Commutative regular incline is distributive lattice and $DL = £$, Theorem (2.4.19) reduces to the following:

Corollary 2.4.27

Let *£* be a distributive lattice whose elements are all linearly ordered. Then *£* is an integral incline.

Theorem 2.4.28 (Theorem (4.1) of [19])

Let *£* be an incline. Then *£* is a integral incline if and only if $G_n(£) = P_n(£)$ for any n $(n \geq 2)$.

Theorem 2.4.29

Let *£* be an incline whose idempotent elements are linearly ordered. Then *DL* is an integral incline $\Leftrightarrow G_n(DL) = P_n(DL)$.

Proof

Since idempotent elements of *£* are linearly ordered and by Lemma (2.3.8), *1* is the multiplicative identity for *DL*, this equivalence follows from Theorem (2.4.28).

Remark 2.4.30

For a regular incline, by Lemma (2.3.8), *1* is the multiplicative identity for *£,* Theorem (2.4.29), reduces to the following result found in [19].

Corollary 2.4.31

If *£* is a regular incline whose elements are all linearly ordered then *£* is an integral incline $\Leftrightarrow$ $G_n(£) = P_n(£)$ for any n $(n\geq 2)$.

Corollary 2.4.32

If *£* is a distributive lattice whose elements are all linearly ordered then *£* is an integral incline $\Leftrightarrow$ $G_n(£) = P_n(£)$ for any n $(n\geq 2)$.

Remark 2.4.33

In Corollary (2.4.31) and Corollary (2.4.32) the condition that the elements are to be linearly ordered cannot be relaxed. This is illustrated in the following:

Example 2.4.34

Let us consider the incline $£ = (\mathcal{P}(D), \cup, \cap)$ in Example (2.3.16).

In this incline the elements are all idempotent but not linearly ordered. Since by Proposition (1.2.20), *£* is regular incline and hence a distributive lattice whose elements are not comparable. For instance the non zero elements $\{a,c\}$ are not comparable, but forms an integral pair, since $\{a,c\} \cap \{b\} = \phi$ and

$\{a,c\} \cup \{b\} = \{a,b,c\} = D$. Hence £ is not an integral incline and from the matrix A in Example (2.4.15), $G_3(£) \neq P_3(£)$, A is invertible matrix of order 3 but not a permutation matrix.

Thus Corollary (2.4.31) and (2.4.32) fails.

Corollary 2.4.35

If £ is not an integral incline then $G_n(£) \neq P_n(£)$, for any $n(n \geq 2)$.

Proof

This follows directly from Corollary (2.4.31).

2.5. Range Symmetric Incline Matrices

In this section, we shall introduce a special type of incline matrix called range symmetric incline matrix for which row and column ranks are equal. This is a generalization of invertible matrices.

Definition 2.5.1

$A \varepsilon £_n$ is a range symmetric incline matrix if and only if $\mathcal{R}(A) = \mathcal{R}(A^T)$. Let $\mathcal{R}(£_n)$ be the set of all range symmetric matrices in $£_n$.

For an invertible matrix $A \varepsilon £_n$, $\mathcal{R}(A) = \mathcal{R}(A^T) = £^n$. Hence A is range symmetric.

Lemma 2.5.2

For $A, B \varepsilon £_n$ and a permutation matrix P, $\mathcal{R}(A) = \mathcal{R}(B)$ if and only if $\mathcal{R}(PAP^T) = \mathcal{R}(PBP^T)$.

Proof

Let $\mathcal{R}(A) = \mathcal{R}(B)$. Then by Lemma (1.2.9),

$\mathcal{R}(AP^T) = \mathcal{R}(A)P^T = \mathcal{R}(B)P^T = \mathcal{R}(BP^T)$.

Let $y \varepsilon \mathcal{R}(PAP^T)$

$\Rightarrow$ $y = x(PAP^T)$ for some $x \varepsilon £^n$

$\Rightarrow$ $y = wAP^T$, where $w = xP$

$\Rightarrow$ $y \, \varepsilon \, \mathcal{R}(AP^T) = \mathcal{R}(BP^T)$

$\Rightarrow$ $y = uBP^T$ for some $u \, \varepsilon \, £^n$.

$\Rightarrow$ $y = (uP^T)PBP^T$ (by Definition (1.2.30))

$\Rightarrow$ $y = zPBP^T$ for some $z \, \varepsilon \, £^n$.

$\Rightarrow$ $y \, \varepsilon \, \mathcal{R}(PBP^T)$

$\Rightarrow$ $\mathcal{R}(PAP^T) \subseteq \mathcal{R}(PBP^T)$

Similarly, $\mathcal{R}(PBP^T) \subseteq \mathcal{R}(PAP^T)$. Hence $\mathcal{R}(PAP^T) = \mathcal{R}(PBP^T)$.

Conversely, let $\mathcal{R}(PAP^T) = \mathcal{R}(PBP^T)$

$\mathcal{R}(A) = \mathcal{R}(P^T(PAP^T)P)$ (by Definition(1.2.30))

$= \mathcal{R}(P^T(PBP^T)P)$

$= \mathcal{R}(B)$ (by Definition (1.2.30))

Hence, $\mathcal{R}(A) = \mathcal{R}(B)$.

Theorem 2.5.3

Let $£$ be an incline. For $A \, \varepsilon \, £_n$, the following statements are equivalent:

(i) A is range symmetric and $\rho(A) = r$

(ii) $\mathcal{C}(A) = \mathcal{C}(A^T)$

(iii) $A^T = AH = KA$ for some incline matrices H, K and $\rho(A) = r$

(iv) PAP^T is range symmetric matrix of rank r for some permutation matrix P.

Proof

(i) $\Leftrightarrow$ (ii) This follows from the Definition (2.5.1) and from $\mathcal{C}(A) = \mathcal{R}(A^T)$ and $\mathcal{C}(A^T) = \mathcal{R}(A)$.

(ii) $\Leftrightarrow$ (iii) This follows directly from Lemma (1.2.8) replacing B by A^T.

(i) $\Leftrightarrow$ (iv) This equivalence follows from Lemma (2.5.2).

Theorem 2.5.4

$A \, \varepsilon \, £_n$ is range symmetric and $\rho(A) = r$ then A can be represented as $A = P\begin{pmatrix} D & DX^T \\ XD & XDX^T \end{pmatrix} P^T$, where P is a permutation matrix, X is a incline matrix and $D \, \varepsilon \, \mathcal{R}(£_r)$ with $\rho(A) = \rho(D) = r$.

Proof

Since $\mathcal{R}(A) = \mathcal{R}(A^T)$, if the ith row A_{i*} is linearly dependent on some rows of A, then the corresponding ith column A_{*i} is also linearly dependent on the corresponding columns of A. Hence by permuting the rows and columns of A, we can bring the r linearly independent rows of A to the top and the corresponding r linearly independent columns of A to the right, so that the normal form of A is $P\begin{pmatrix} D & DX^T \\ XD & XDX^T \end{pmatrix} P^T$, where D is $r \times r$ matrix with no zero rows and no zero columns and $\mathcal{R}(D) = \mathcal{R}(D^T)$. X is a matrix over an incline that expresses the dependence of the last $(n\text{-}r)$ rows on the r linearly independent rows of A. $\rho(A) = \rho(D) = r$.

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CHAPTER 3
REGULAR MATRICES OVER AN INCLINE

We derive equivalent conditions for a matrix over an incline whose idempotent elements are linearly ordered to be regular, which include as special cases for matrices over a regular incline whose elements are all linearly ordered and for matrices over a distributive lattice whose elements are all linearly ordered and as a generalization of the results on fuzzy matrices found in [13]. We have discussed the regularity of matrices over an incline interms of full row space, full column space factorization and studied the properties of g-inverses of a matrix over *DL.* An algorithm is given in [27] to determine the regularity and finding a g-inverse for matrices over an incline in which idempotent elements are linearly ordered. Here, we provide an algorithm for matrices over a regular incline $\pounds$ whose elements are not linearly ordered and illustrated with suitable examples.

3.1. REGULAR MATRICES

In this section, we derive some equivalent conditions for a matrix over an incline whose idempotent elements are linearly ordered to be regular. In [28], it is proved that a matrix over the max-min Fuzzy algebra $\mathcal{F} = [0,1]$ is regular if and only if it is a retract of $\mathcal{F}^n$. Here we shall extend this for a matrix over an incline.

Just for sake of completeness we shall state the following Definition for a matrix over an incline.

Definition 3.1.1

A matrix $A \ \varepsilon \ £_{mn}$ is regular if and only if there exists a matrix $X \ \varepsilon \ £_{nm}$ such that $AXA = A$, $X \ \varepsilon A\{1\}$ and $A \ \{1\}$ denotes the set of all g-inverses of A.

Theorem 3.1.2

Let $£$ be an incline. For a regular matrix $A \ \varepsilon \ £_{mn}$ the following conditions are hold:

(i) There exists an idempotent matrix $E \ \varepsilon \ £_n$ such that $\mathcal{R}(E) = \mathcal{R}(A)$

(ii) $\mathcal{R}(A)$ is a retract of $£^n$

(iii) There exists an idempotent matrix $F \ \varepsilon \ £_m$ such that $\mathcal{C}(A) = \mathcal{C}(F)$.

Proof

(i) Since A is regular by Definition (3.1.1), $A= AXA$ for some $X \ \varepsilon \ £_{nm}$. By Lemma (1.2.9), we have $\mathcal{R}(A) = \mathcal{R}(AXA) \subseteq \mathcal{R}(XA) \subseteq \mathcal{R}(E) \subseteq \mathcal{R}(A)$. Therefore, $\mathcal{R}(A) = \mathcal{R}(E)$, where $E=XA \ \varepsilon \ £_n$ is an idempotent matrix. Thus (i) holds.

(ii) Again by Lemma (1.2.9) and by Proposition (1.2.27), we have $\mathcal{R}(E) = \mathcal{R}(E^2) = \mathcal{R}(E)E = \mathcal{R}(A)E \subseteq (£^n) E = \mathcal{R}(E) \Rightarrow \mathcal{R}(A) = £^n(E)$. Then, $\mathcal{R}(A)$ is a retract of $£^n$ follows from the Definition (1.2.26). Thus (ii) holds.

(iii) It can be proved in the same manner by choosing the idempotent matrix $F=AX$ in the regularity equation $A=AXA$.

Remark 3.1.3

In general, the converse need not be true, (refer Example (3.1.6)). However, the converse holds under certain conditions. This is given in the following:

Theorem 3.1.4

Let £ be an incline whose idempotent elements are all linearly ordered and forms a vector space over an incline £. For $A \varepsilon DL_{mn}$, the following conditions are equivalent:

(i) A is regular

(ii) There exists an idempotent matrix $E \varepsilon £_n$ such that $\mathcal{R}(E) = \mathcal{R}(A) = (DL^n)E$

(iii) $\mathcal{R}(A)$ is a retract of $£^n$

(iv) There exists an idempotent matrix $F \varepsilon £_m$ such that $\mathcal{C}(A) = \mathcal{C}(F) = (DL^n)F$. In either case, $\rho_r(A) = \rho_c(A)$.

Proof

(i) $\Rightarrow$ (ii) The existence of an idempotent matrix E with $\mathcal{R}(A) = \mathcal{R}(E)$ has been proved in Theorem (3.1.2). Under the condition that the idempotent elements are linearly ordered, for any $x,y \varepsilon DL^n$, $x+y \varepsilon DL^n$ and $0 \varepsilon DL^n$. Therefore DL^n is a subspace of $£^n$. Since $A \varepsilon DL_{mn}$, each row of $A \varepsilon DL^n$ and DL forms a vector space over £ hence $\mathcal{R}(A)$, the space spanned by the rows of A is the subspace of DL^n. Therefore, $\mathcal{R}(E)=\mathcal{R}(A)\subseteq DL^n$ for the idempotent matrix E. Further, by Lemma (1.2.9), we have $\mathcal{R}(E)= \mathcal{R}(E^2)=\mathcal{R}(E)E=\mathcal{R}(A)E\subseteq(DL^n)\, E \subseteq(£^n)E= \mathcal{R}(E)$. Hence $\mathcal{R}(A) = \mathcal{R}(E) = (DL^n)E$. Thus (ii) holds.

(ii) $\Rightarrow$ (iii) From (ii) we get, $\mathcal{R}(A)=\mathcal{R}(E) = (DL^n)E=(£^n)E$. Then, $\mathcal{R}(A)$ is a retract of $£^n$ follows from the Definition (1.2.26). Thus (iii) holds.

(iii) $\Rightarrow$ (i) Since $\mathcal{R}(A)$ is a retract of $£^n$, by Definition (1.2.26), $\mathcal{R}(A)=(£^n)\,E=\mathcal{R}(E)$. By Lemma (1.2.8), $\mathcal{R}(E)\subseteq\mathcal{R}(A)$ implies $E=XA$ for some $X\ \varepsilon\ £_m$. Since E is regular being idempotent, E itself is one choice of g-inverse of A. Therefore, by Lemma (1.2.10), $\mathcal{R}(A)\subseteq\mathcal{R}(E)\Rightarrow A=AE^{-}E=AE=AXA$, hence A is regular. Thus (i) holds.

(ii) $\Leftrightarrow$(iv) This equivalence can be proved in the same manner by choosing the idempotent matrix $F=AX$ in the regularity equation $A=AXA$.

For $A\ \varepsilon\ DL_{mn}$ is regular then $\rho_r(A)=\rho_c(A)$ follows from the fact that elements of DL are linearly ordered [27].

Remark 3.1.5

$A\ \varepsilon\ DL_{mn}\Rightarrow\mathcal{R}(A)\subseteq DL^n$ but not conversely. In the above Theorem (3.1.4), the conditions that $A\ \varepsilon\ DL_{mn}$ and forms a vector space over £ are essential. These are illustrated in the following examples:

Example 3.1.6

Let us consider the incline $£=\{0, a, b, c, d, 1\}$ in Example (2.1.8).

$£$ is a commutative incline which is not a distributive lattice. In this incline, for $x,y\ \varepsilon\ £$, $xy=0$ (or) d. Here 0 and d are the only idempotent elements, that is, $DL=\{0,d\}$; 0 and d are comparable. For $\alpha\ \varepsilon\ £$, $x\ \varepsilon\ DL$, $\alpha\, x\ \varepsilon\ DL$. Hence DL forms a vector space over $£$.

Let $A=\begin{pmatrix} b & 1\\ 0 & c\end{pmatrix}\ \varepsilon\ £_2$ then for any $X=\begin{pmatrix} x & y\\ u & v\end{pmatrix}\ \varepsilon\ £_2$

(11)th entry of $A\,X\,A$ is $(bx+1u)\,b\neq b$.

Therefore, $A\,X\,A\neq A$, A is not regular.

Now, let us take $X=\begin{pmatrix} 1 & b\\ a & d\end{pmatrix}\ \varepsilon\ £_2$ then XA $=\begin{pmatrix} d & d\\ 0 & d\end{pmatrix}=E$.

$E^2=\begin{pmatrix} d & d\\ 0 & d\end{pmatrix}=E\ \varepsilon\ DL_2$ and

$\mathcal{R}(E)=\mathcal{R}(A)=\{(0,0), (0,d), (d,0),(d,d)\}=DL^2$. Thus (ii) and (iii) hold. Here $A\notin DL_2$ and A is not regular.

Hence the Theorem (3.1.4) fails.

Example 3.1.7

Consider the incline $£ = \{[0,1],+,\cdot\}$ in Example (2.3.4). Here, the idempotent elements are 0 and 1, that is, $DL = \{0,1\}$ and they are comparable. In this incline, 1 is the multiplicative identity as well as the greatest element.

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \varepsilon\, DL_2$

A is regular being idempotent and $\mathcal{R}(A) = \{\lambda(1,1) + \mu(0,1)/\ \lambda,\mu\ \varepsilon[0,1]\} = \{(a,b)/0 \le a \le b \le 1\}$. But $(DL^2)\ E \ne \mathcal{R}(A)$, for any idempotent matrix $E\ \varepsilon\ £_2$. Thus (ii) fails. For $x\ \varepsilon\ DL,\ \alpha\ \varepsilon\ £,\ \alpha x \notin DL$. Hence DL does not form a vector space over the incline $£$.

Now we deduce the result of the Theorem (3.1.4) for the following special types of incline such as regular incline, distributive lattice and an integral incline.

Corollary 3.1.8

Let $£$ be a regular incline whose elements are all linearly ordered and $A\ \varepsilon\ £_{mn}$ then the following are equivalent:

(i) A is regular

(ii) There exists an idempotent matrix $E\ \varepsilon\ £_n$ such that $\mathcal{R}(E) = \mathcal{R}(A) = (£^n)E$

(iii) $\mathcal{R}(A)$ is a retract of $£^n$

(iv) There exists an idempotent matrix $F\ \varepsilon\ £_m$ such that $\mathcal{C}(A) = \mathcal{C}(F) = (£^m)F$. In either case, $\rho_r(A) = \rho_c(A)$.

Proof

Since $£$ is regular incline by Proposition (1.2.20), each element of $£$ is idempotent and therefore $DL = £$. Then the rest follows from Theorem (3.1.4) under the condition that all elements are linearly ordered.

Remark 3.1.9

Since by Proposition (1.2.21), a commutative regular incline is a distributive lattice and $DL = £$, Theorem (3.1.4) reduces to the following.

Corollary 3.1.10

Let $£$ be a distributive lattice whose elements are all linearly ordered and $A \,\varepsilon\, £_{mn}$ then the following are equivalent:

(i) A is regular

(ii) There exists an idempotent matrix $E \,\varepsilon\, £_n$ such that $\mathcal{R}(E) = \mathcal{R}(A) = (£^n)\, E$

(iii) $\mathcal{R}(A)$ is a retract of $£^n$

(iv) There exists an idempotent matrix $F \,\varepsilon\, £_m$ such that $\mathcal{C}(A) = \mathcal{C}(F) = (£^m)\, F$. In either case, $\rho_r(A) = \rho_c(A)$.

Remark 3.1.11

The linearity condition on elements of $£$ in Corollary (3.1.8) and Corollary (3.1.10) cannot be relaxed. This is illustrated in the following example:

Example 3.1.12

Let us consider the incline $£ = (\mathcal{P}(D), \cup, \cap)$ in Example (2.3.16).

Let $A = \begin{pmatrix} \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \{c\} \end{pmatrix} \varepsilon\, £_3$

and $X = \begin{pmatrix} \{a\} & \{b\} & \phi \\ \{b\} & \{a\} & \phi \\ \phi & \phi & \{c\} \end{pmatrix} \varepsilon\, £_3$

On computation we get $A\,X\,A = A$. Therefore A is a regular matrix. Here, $£$ is commutative regular incline whose elements are all idempotent under the set intersection as the multiplication operation. By Proposition (1.2.21), $£$ is a distributive lattice. The elements of $£$ are all idempotent but not linearly ordered. For instance, $\{a\}$, $\{b\}$ and $\{c\}$ are not comparable. For A, $A_{1*} = A_{2*} = \{a,b\}\ A_{3*}$. $\rho_r(A)=1$. All the columns are linearly independent, hence $\rho_c(A) = 3$. Therefore, $\rho_r(A) \neq \rho_c(A)$. The equality condition of row rank and column rank for a regular matrix in Corollary (3.1.8) and Corollary (3.1.10) fails.

Remark 3.1.13

For a regular incline whose elements are all linearly ordered then the incline operations reduces to the max - min composition, by Lemma (2.3.13). Therefore, Corollary (3.1.8) reduces to the result on regularity of fuzzy matrix due to Kim and Roush [28].

Corollary 3.1.14

Let A be matrix over the max-min Fuzzy algebra then the following conditions are equivalent:

(i) A is regular

(ii) There exists an idempotent fuzzy matrix $H \varepsilon £_n$ such that $\mathcal{R}(H) = \mathcal{R}(A)$

(iii) $\mathcal{R}(A)$ is a retract of V^n

(iv) There exists an idempotent matrix $H \varepsilon £_m$ such that $\mathcal{C}(A) = \mathcal{C}(H)$. In either case, $\rho_r(A) = \rho_c(A)$.

Theorem 3.1.15

Let $£$ be an incline whose idempotent elements are linearly ordered. If $A \varepsilon DL_{mn}$ is a regular matrix then the row rank, the column rank and the factor rank of A are equal.

Proof

First we prove that $\rho_t(A) = \rho_r(A)$.

Let r be the row rank and t be the factor rank of A. If $\mathcal{R}(A)$ can be generated by the rows of an $r\times n$ matrix M, then there is an $n \times r$ incline matrix L such that $A = LM$. Thus $r \geq t$ and we will show $r \leq t$ in follows. Since $A \varepsilon DL_{mn}$ is regular, let $AXA = A$ for some $X \varepsilon £_{nm}$ and let $B=XA$. By Lemma

(1.2.8), it follows that $\mathcal{R}(B)\subseteq \mathcal{R}(A)$. Since $AB = AXA=A$, $\mathcal{R}(A)\subseteq \mathcal{R}(B)$. Thus $\mathcal{R}(A) = \mathcal{R}(B)$. By Definition (1.2.4), $\rho_r(A) = \rho_r(B)$. Since the factor rank of A is t, there exists $C \varepsilon £_{mt}$ and $D \varepsilon £_{tn}$ such that $A = CD$. Since $B=XA= XCD= ED$, where $E=XC$, t is the smallest integer and $\mathcal{R}(A) = \mathcal{R}(B)$, therefore $\rho_f(A) = \rho_f(B)$. Let $B[0] = B=(b_{ij})$ and for each integer $1 \leq k \leq n$, let $B[k]$ be the matrix obtained from $B[k\text{-}1]$ by replacing the kth row $B[k\text{-}1]_{k*}$ of $B[k\text{-}1]$ with $b_{kk}\ B[k\text{-}1]_{k*}$. Then by Theorem (2.3.31), $B[k]^2 = B[k]$ for each $1 \leq k \leq n$. Choose an $n \times n$ permutation matrix P so that the first r-many rows of an idempotent matrix $E = (e_{ij}) = PB[n]P^T$ are linearly independent and $e_{ii} \geq e_{ij}$ when $i<j \leq r$. Then we have $e_{ii}\geq e_{ij}$ for each i and j. Since the factor rank of E is also t, there exist $C \varepsilon £_{nt}$ and $D \varepsilon £_{tn}$ such that $E=CD$. Now let $R_t = C_{*i}D_{i*}$ for each $1 \leq i \leq t$. Clearly R_i's are of rank -1.

Among the rank - 1 matrices $R_i's$, let there be an integer k such that $F= (f_{ij}) = R_k$ satisfies $f_{uu} =e_{uu}$ and $f_{vv} = e_{vv}$ for some u and v with $u < v \leq r$. Then $f_{uu} \geq f_{vv}$ *and* $F_{v*} = f_{vv}\ F_{u*}$. Since the row rank of F is one and f_{uu} (f_{vv}) is a maximal entry of $F_{u*}(F_{v*})$ respectively. Thus f_{vu} must be f_{vv} and $f_{uu}\geq f_{uv} \geq f_{vv}$ holds. Since $f_{vv}= e_{vv}\geq e_{vu}\geq f_{vu}$ and $f_{vu}=f_{vv}$ we have $e_{vu}=e_{vv}$. Since $e_{uu}=f_{uu}\geq e_{uv}\geq f_{uv}$ and $f_{uv} \geq f_{vv}=e_{vv}$. We have $e_{uu}\geq e_{uv}\geq e_{vv}$. By using $E=E^2$, we can derive $e_{uu}E_{u*} +e_{uv}\ E_{v*} \leq E_{u*}$ and $e_{vv}\ E_{u*}\ e_{vv}\ E_{v*} \leq E_{v*}$. Therefore $e_{uv}E_{v*} \leq E_{u*}$ and $e_{vu}\ E_{u*} \leq E_{v*}$ hold and from this we have $e_{vv}\ e_{uv}E_{u*} \leq e_{vv}\ E_{u*}$ and $e_{vv}e_{vu}\, E_{u*}\leq e_{vv}E_{v*}$. Thus $E_v\ e_{vv}E_{u*}$ and $e_{vv}\ E_{u*} \leq E_{v*}$ hold, this means $E_{v*} = e_{vv}E_{u*}$. This is a contradiction, since the first r-many rows of E are linearly independent. Therefore, when summing up the t-many rank $^-1$ matrices $R_i's$, each R_j can make at most one diagonal entry $e_{kk}(1 \leq k \leq r)$. Thus $t \geq r$. Hence $r=t$.

Now consider the transpose A^T of A. Then A^T is a regular matrix and the row of A^T is the column rank of A. Since the factor rank of A is equal to that of A^T, by the above reasoning the column rank of A is equal to the factor rank of A. Therefore the row rank, the column rank and the factor rank of A are equal.

3.2. Properties of g-Inverses

In this section, we derive some basic properties of generalized inverse of a regular matrix over an incline $£$.

Proposition 3.2.1

Let $£$ be incline. Let $A \varepsilon £_{mn}$ be a regular matrix, X be a g-inverse of A. Then

(i) $X^T \varepsilon A^T \{1\}$

(ii) $\lambda X \varepsilon (\lambda A)\{1\}$ for $\lambda \varepsilon DL$

(iii) If P and Q are permutation matrices, then $Q^T X P^T \varepsilon PAQ \{1\}$.

Proof

(i) Let $A \varepsilon £_{mn}$ be regular and X be a g-inverse of A then by Definition (3.1.1), $AXA = A$. Taking the transpose on both sides, we get $A^T X P^T = A^T$. Thus $X^T \varepsilon A^T \{1\}$.

(ii) Let $A = (a_{ij}) \varepsilon £_{mn}$. Then $\lambda A = \lambda(a_{ij}) = A\lambda$. Since $\lambda\lambda = \lambda$ for $\lambda \varepsilon DL$ Therefore $(\lambda A)(\lambda X)(\lambda A) = \lambda A$. Thus $\lambda X \varepsilon (\lambda A)\{1\}$.

(iii) Since P and Q are permutation matrices, P and Q are invertible, $P^- = P^T$, $Q^- = Q^T$. Let $AXA = A$.

$$\begin{aligned}(PAQ)(Q^T X P^T)(PAQ) &= PA(QQ^T)X(PP^T)AQ \\ &= PA\,X\,AQ \\ &= P(A\,X\,A)Q \\ &= PAQ\end{aligned}$$

Thus $Q^T X P^T \varepsilon PAQ \{1\}$.

Just for sake of completeness we shall state the properties of a regular matrix over DL_{mn} in the following, that can be proved along the same lines as that for fuzzy matrices [36].

Proposition 3.2.2

Let $£$ be an incline whose idempotent elements are all linearly ordered. Let $A \varepsilon DL_{mn}$ be regular and X be a g-inverse of A. Then

(i) AX and XA are idempotent, $\mathcal{R}(A) = \mathcal{R}(XA)$, $\mathcal{C}(A) = \mathcal{C}(AX)$ and have the same rank as A

(ii) $\rho(X) \geq \rho\ (A)$.

Proof

(i) forms part of Theorem (3.1.4).

(ii) Since $\rho(AX) \leq min\ \{\rho(A), \rho(X)\}$ it follows that $\rho(A) \leq \rho\ (X)$.

Theorem 3.2.3

Let $\pounds$ be an incline. For A, $B\ \varepsilon\ \pounds_{mn}$ with $\mathcal{R}(A) = \mathcal{R}(B)$ *(or)* $\mathcal{C}(A) = \mathcal{C}(B)$. Then A is a regular matrix $\Leftrightarrow B$ is a regular matrix.

Proof

Suppose $A\ \varepsilon\ \pounds_{mn}$ is regular and $\mathcal{R}(A) = \mathcal{R}(B)$. Then by Lemma (1.2.8), $\mathcal{R}(A) \subseteq \mathcal{R}(B) \Leftrightarrow A= XB$ for some $X\ \varepsilon\ \pounds_{m}$. Since A is regular, by Lemma (1.2.10), $\mathcal{R}(B) \subseteq \mathcal{R}(A) \Leftrightarrow B=B\ A^{-}A$ for all $A^{-} \varepsilon\ A\{1\}$. Hence $B=B\ A^{-}A=BA^{-}(XB)=\ B\ (A^{-}X)\ B$. Thus B is a regular matrix.

Converse can be proved by interchanging A by B in the preceding proof.

Under the condition $\mathcal{C}(A) = \mathcal{C}(B)$, the Theorem can be proved in a similar manner and hence omitted.

Remark 3.2.4

In general, for an incline matrix A, there is no relation between the regularity of AA^T, A^TA and A. Here, we shall see the condition for their equivalence in the following:

Theorem 3.2.5

Let $\pounds$ be an incline

(i) If $A\ \varepsilon\ \pounds_{mn}$ with $\mathcal{R}(A) = \mathcal{R}(A^TA)$, then A^TA is a regular matrix $\Leftrightarrow A$ is a regular matrix

(ii) If $A \varepsilon \pounds_{mn}$ with $\mathcal{C}(A) = \mathcal{C}(AA^T)$, then AA^T is a regular matrix $\Leftrightarrow A$ is a regular matrix.

Proof

(i) Follows from Theorem (3.2.3) by replacing B by A^TA.

(ii) Follows from Theorem (3.2.3) by replacing B by AA^T.

Remark 3.2.6

In Proposition (3.2.1), the condition $\lambda \varepsilon DL$ is essential. This is illustrated in the following example:

Example 3.2.7

Consider the incline $\pounds = \{[0,1], +, \cdot\}$ in Example (2.3.4).

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \varepsilon \pounds_2$ is regular being idempotent.

$X = A$ itself is a g-inverse of A. Here, $DL = \{0,1\}$.

For $\lambda = 0.5 \notin DL$,

$\lambda A = \begin{pmatrix} 0.5 & 0.5 \\ 0 & 0.5 \end{pmatrix}$. But $(\lambda A)(\lambda A)(\lambda A) \neq (\lambda A)$. Therefore $\lambda A \notin A\{1\}$.

Remark 3.2.8

In Theorem (3.2.5), the conditions on row spaces and column spaces are essential can be seen by the following example:

Example 3.2.9

Let us consider the incline $\pounds = \{0, a, b, c, d, 1\}$ in Example (2.1.8).

In this incline, for $x, y \varepsilon \pounds$, $xy = 0$ (or) d. Hence 0 and d are the only idempotent elements, that is, $DL = \{0, d\}$ and elements of DL are linearly ordered. Therefore, $\pounds$ is an incline with idempotent elements are linearly ordered.

Let $A = \begin{pmatrix} b & 1 \\ c & 0 \end{pmatrix} \varepsilon \pounds_2$

$A^TA = \begin{pmatrix} b & c \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & 1 \\ c & 0 \end{pmatrix} = \begin{pmatrix} d & d \\ d & d \end{pmatrix} = AA^T$

is regular in $\pounds_2$ being idempotent.

But for any $X = \begin{pmatrix} x & y \\ u & v \end{pmatrix} \varepsilon\, \pounds_2$, $A\,X\,A \neq A$.

For instance, (11)th entry of $A\,X\,A$ is $(bx+1u)b+(by+1v)c \neq b$.

Therefore $A\,X\,A \neq A$ hence A is not regular.

$\mathcal{R}(A) = \{(0,0), (d,0), (0,d), (d,d)\} = DL^2$.

$\mathcal{R}(A^TA) = \{(0,0), (d,d)\} = \mathcal{R}(AA^T)$

Therefore, $\mathcal{R}(A) \neq \mathcal{R}(A^TA)$ and $C(A) \neq C(AA^T)$.

Thus Theorem (3.2.5) fails.

3.3. Regularity Algorithm

In this section, we shall discuss the regularity of an incline matrix using full space factorization as an extension of that for fuzzy matrices found in [28] and we shall provide an algorithm to determine the regularity of incline matrices.

Theorem 3.3.1

Let $A = FG$ be a full row space factorization. Then G is a regular matrix $\Leftrightarrow A$ is a regular matrix.

Proof

Since $A= FG$ is a full row space factorization, by Definition (2.2.18), $\mathcal{R}(A) = \mathcal{R}(G)$. Then from Theorem (3.2.3) it follows that A is a regular matrix $\Leftrightarrow G$ is a regular matrix.

Theorem 3.3.2

Let $A= FG$ be a full column space factorization. Then F is a regular matrix $\Leftrightarrow A$ is a regular matrix.

Proof

This can be proved in the same manner as that of Theorem (3.3.1) and hence omitted.

By combining Theorem (3.3.1) and Theorem (3.3.2), we have the following:

Theorem 3.3.3

Let $A=FG$ be a full space factorization. Then A is a regular matrix $\Leftrightarrow G$ is a regular matrix $\Leftrightarrow F$ is a regular matrix.

Remark 3.3.4

Full Space factorization condition on A in Theorem (3.3.3) is essential. This is illustrated in the following example:

Example 3.3.5

Let us consider the matrices over an incline $\pounds = \{0,a,b,c,d,1\}$ in Example (2.1.8).

For $A = \begin{pmatrix} d & d \\ d & d \end{pmatrix} \varepsilon\, \pounds_2$ $\quad B= \begin{pmatrix} 1 & 0 \\ d & 0 \end{pmatrix} \varepsilon\, \pounds_2$ and

$C = \begin{pmatrix} b & c \\ 0 & 0 \end{pmatrix} \varepsilon\, \pounds_2$. In Example (2.2.11), we have seen that $A=BC$ is not a full space factorization. Here A is regular being idempotent but $B= \begin{pmatrix} 1 & 0 \\ d & 0 \end{pmatrix}$ and $C= \begin{pmatrix} b & c \\ 0 & 0 \end{pmatrix}$ are not regular and Theorem (3.3.3) fails.

Remark 3.3.6

Every idempotent matrix is regular but not the converse. This can be seen from the following example:

Example 3.3.7

Consider an incline $\pounds = \{[0,1], sup\,(x,y), min\,(x,y)\}$

$A = \begin{pmatrix} 0.5 & 1 \\ 1 & 0.2 \end{pmatrix} \varepsilon\, \pounds_2$

By applying the regularity algorithm from [27], A is regular, since each row contains a maximal element 1, which is idempotent and they form a permutation matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Here, A is not idempotent,

$A^2 = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix} \neq A.$ Hence the converse need not be true.

Example 3.3.8

Let us consider the incline $\pounds = \{[0,1], +, \cdot\}$ in Example (2.3.4). Let $A = \begin{pmatrix} 0.5 & 0.2 \\ 0.6 & 1 \end{pmatrix} \varepsilon\ \pounds_2$. Here A is not regular since the maximal element in the first row is 0.5, which is not idempotent $((0.5)^2 \neq 0.5)$, under the usual multiplication. Hence A is not idempotent.

Remark 3.3.9

Every invertible matrix is regular, but the converse is not true. This can be seen from the following example:

Example 3.3.10

Let us consider the incline $\pounds$ and the matrix A = $\begin{pmatrix} 0.5 & 1 \\ 1 & 0.2 \end{pmatrix}$ $\varepsilon\ \pounds_2$ in Example (3.3.7) and we have seen that A is regular. Here, $AA^T \neq A^TA = I_n$. Therefore A is not an invertible matrix.

ALGORITHM

An algorithm is given in [27] to determine the regularity and finding a g-inverse for matrices over an incline in which idempotent elements are linearly ordered. In step 1, one has to check the maximal element row by row, which cannot be applied when elements are not comparable. In this part, we provide an algorithm for matrices over a regular incline $\pounds$ whose elements are all not linearly ordered. Further, if A is regular whose non zero rows form a standard basis then there must be a g-inverse which is a permutation matrix in the case that, idempotent elements are linearly ordered in the original incline [11]. We observe that this fails for an incline which is not regular. This is illustrated in the following example:

Example 3.3.11

Let us consider the matrices over the incline $\pounds = \{0,a,b,c,d,1\}$ in Example (2.1.8),here $\pounds$ is not a regular incline and 1 is not the multiplicative identity for $\pounds$ and $\pounds_n$ has no permutation matrix.

For instance, $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \varepsilon\ \pounds_2$ and $P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \varepsilon\ \pounds_2$

are not permutation matrices in $\pounds_2$.

$P_1P_1^T = P_1^TP_1 = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \neq I_2$ and $P_2P_2^T = P_2^TP_2 = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \neq I_2$

However, $E = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$ is regular being idempotent and $EP_2E=E$.

Rows of E forms a standard basis for $\mathcal{R}(E)$.

Definition 3.3.12

A standard basis $\{c_1,c_2,\ldots,c_n\}$ is said to be orthogonal basis if $c_ic_j^T=0$, for all $i \neq j$ and orthonormal basis if $c_i\, c_j^T=0$, for i $\neq$j and $c_ic_i^T=1$. Where 0 and 1 are the least and greatest element of the incline $\pounds$.

Example 3.3.13

In the max-min Fuzzy algebra,

$c_1= (0.5, 0.5, 0.5)$, $c_2 = (0, 1, 0.5)$, $c_3 =(0, 0.5, 1)$ is a standard basis for $\mathcal{F}^3$ [28]. It is not an orthogonal standard basis, since $c_1c_2^T \neq 0$ and it is not an orthonormal standard basis, since $c_1c_1^T=0.5 \neq 1$.

Example 3.3.14

In Example (2.3.16), $\{(\{a\}, \phi,\phi), (\phi,\phi,\{c\}), (\phi, \{b\},\phi)\}$ is a standard basis for the subspace. It is orthogonal but not orthonormal.

We shall modify Theorem (1.2.31) for a matrix over a regular incline in the following way:

Theorem 3.3.15

Let £ be a regular incline and $A \varepsilon £_{mn}$, whose non zero rows forms an orthogonal standard basis and each basis vector has exactly one non zero entry. Then A has a g - inverse which is a permutation matrix.

Proof

Let E be a matrix whose rows are the standard basis. Let $\{c_1, c_2, \ldots c_n\}$ be the orthogonal standard basis for $\mathcal{R}(E) = \mathcal{R}(A)$.

Let $E = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$, $E^T = (c_1^T, c_2^T, \ldots, c_n^T)$. Since $\{c_1, c_2, \ldots. c_n\}$ forms an orthogonal standard basis, $c_i c_j^T = 0$ for $i \neq j$ and $c_i c_i^T \neq 0$ for each $i = 1$ to n. Hence the rows of E are orthogonal. Since each row of E has exactly one non zero entry. If $E = (c_{ij})$ and $c_{ij} \neq 0$ for some i and j then $c_{is} = 0$ for all $s \neq j$ and $c_{rj} = 0$ for all $r \neq i$. Define the matrix P such that

$$P_{ij} = \begin{cases} 1 \text{ if } c_{ij} \neq 0 \\ 0 \text{ Otherwise} \end{cases}$$

Clearly P is a permutation matrix and $EP = Y$, where Y is the diagonal matrix whose ith diagonal entry is the non zero entry in the ith row of E.

$$\text{Further, } EE^T = \begin{pmatrix} \sum_k c_{1k}\, c_{1k} & 0 & \ldots & 0 \\ 0 & \sum_k c_{2k}\, c_{2k} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & \sum_k c_{nk}\, c_{nk} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_k c_{1k}^2 & 0 & \ldots & 0 \\ 0 & \sum_k c_{2k}^2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & \sum_k c_{nk}^2 \end{pmatrix}$$

Since £ is regular, by Proposition (1.2.19), $c_{ik}^2 = c_{ik}$ for each i.

$$EE^T = \begin{pmatrix} \sum_k c_{1k} & 0 & .. & 0 \\ 0 & \sum_k c_{2k} & .. & 0 \\ : & : & : & : \\ 0 & 0 & & \sum_k c_{nk} \end{pmatrix} = X,$$

Where X is the diagonal matrix whose ith diagonal entry is $\sum c_{ik}$ = the sum of the entries of the ith row of E. Hence $X = Y$ and $EE^T = X = Y = EP$. By regularity of £, we have $E = YP^T \Rightarrow EE^TE = YP^T\ PY^T\ YP^T = Y(P^TP)\ Y^TYP^T = YY^TYP^T = YP^T = E \Rightarrow EPE = EE^TE = E$. Hence E is regular, P is a g-inverse of E, which is a permutation matrix.

Hence the Theorem.

Algorithm 3.3.16

(To test a matrix A for regularity and when it has a permutation matrix as its g-inverse)

$A\ \varepsilon\ £_{mn}$ in a regular incline £.

Step 1 : Find a standard basis for the row space of A. Let $\{c_1, c_2, \ldots. c_n\}$ be the standard basis for $\mathcal{R}(A)$.

Step 2 : Let E be a matrix whose non zero rows are the members of the orthogonal standard basis.

Step 3 : A is regular $\Leftrightarrow$ E is regular .

Step 4 : A is regular and has a g-inverse, which is a permutation matrix $\Leftrightarrow$ E is regular and has a g-inverse, which is a permutation matrix.

Step 5 : Choose a matrix R such that $RA = E$.

Step 6 : Then PR is a g-inverse of A.

Step 7 : $(PR)\ A\ (PR)$ is a semi inverse of A.

Validity of algorithm 3.3.17

Step 1 : Step 1 is valid by the fact that finite subspace $\mathcal{R}(A)$ has a standard basis.

Step 2 : By construction of E, $\mathcal{R}(A) = \mathcal{R}(E)$.

Step 3 : Step 3 is valid by Theorem (3.2.3), A is a regular matrix $\Leftrightarrow E$ is a regular matrix.

Step 4 : Step 4 is valid by Theorem (3.3.15), permutation matrix P is a g - inverse of $E,$ when the rows of E has a orthogonal standard basis.

Step 5 : Step 5 is valid by Lemma (1.2.8), $E = RA$ for some $R \,\varepsilon\, £_m.$

Step 6 : Step 6 is valid again by Lemma (1.2.8), $A = SE$ for some $S \,\varepsilon\, £_m.$

Step 7 : Step 7 is valid, Let $Y = (PR)\, A\, (PR)$ then $AYA = A$ and $YAY = Y$, therefore Y is a semi inverse of A.

Remark 3.3.18

Stop at step 3 for regularity of the matrix A. Stop at Step 4 for regular, when it has a permutation matrix as its g-inverse. Stop at step 6, to find out g - inverse (or) 1 -inverse of A. Stop at step 7, to find out semi - inverse (or) *{1,2}* inverse of A.

The algorithm is illustrated by the following examples.

Example 3.3.19

Let us consider the incline $£ = (\mathcal{P}(D),\ \cup, \cap)$, in Example (2.3.16). The elements of $£$ are idempotent but not linearly ordered.

Let $A = \begin{pmatrix} \{a\} & \{c\} \\ \phi & \{b\} \end{pmatrix} \varepsilon\, £_2$

Step 1 : For this A, $\mathcal{R}(A) = \{(x,y) / x = \alpha a,\ y = \alpha c + \beta b$, for $\alpha, \beta \,\varepsilon\, £\}$

By applying the incline operatio , we get

$\mathcal{R}(A) = \{(\phi,\phi), (\{a\},\phi), (\phi,\{b\}), (\phi,\{c\}), (\{a\}, \{c\}),(\{a\},\{b\}), (\{a\}, \{b,c\}), (\phi, \{b,c\})\}.$

Single element in $\mathcal{R}(A)$ cannot span $\mathcal{R}(A)$. Here, for the basis vectors $(\{a\},\{c\}),$ the condition fails.

Step 2 : $APA = EPE \neq E = A$ for both permutation matrices

$P_1 = \begin{pmatrix} D & \phi \\ \phi & D \end{pmatrix}$ and $P_2 = \begin{pmatrix} \phi & D \\ D & \phi \end{pmatrix}$. Hence, A has no permutation matrix as its g –inverse.

Example 3.3.20

Let us consider the incline $\pounds = (\mathcal{P}(D), \cup, \cap)$, in Example (2.3.16)

Let $A = \begin{pmatrix} \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \{c\} \end{pmatrix}$

Step 1 : For this A, $\mathcal{R}(A) = \{(\phi,\phi,\phi), (\{a\}, \phi,\phi), (\phi, \{b\},\phi), (\phi,\phi,\{c\}), (\phi,\{b\},\{c\}), (\{a\},\phi,\{c\}), (\{a\},\{b\},\{c\}), (\{a\}, \{b\},\phi)\}$.

$\{(\{a\},\phi,\phi), (\phi,\phi,\{c\}), (\phi,\{b\},\phi)\}$ is a standard basis for $\mathcal{R}(A.)$

Step 2 : Let $E = \begin{pmatrix} \{a\} & \phi & \phi \\ \phi & \phi & \{c\} \\ \phi & \{b\} & \phi \end{pmatrix}$, the rows of E are orthogonal, $c_1 c_1^T = \{a\}$, $c_2 c_2^T = \{c\}$, $c_3 c_3^T = \{b\}$.

$EE^T = \begin{pmatrix} \{a\} & \phi & \phi \\ \phi & \{c\} & \phi \\ \phi & \phi & \{b\} \end{pmatrix} = X.$

Step 3 : E is not an idempotent matrix such that there exists a matrix $P = \begin{pmatrix} D & \phi & \phi \\ \phi & \phi & D \\ \phi & D & \phi \end{pmatrix}$. $EP = Y$, on computation we get $EPE = E$, P is a g-inverse of E, which is a permutation matrix. Hence A is regular.

To find the g-inverse:

Step 4 : Choose $R = \begin{pmatrix} \{a\} & \{c\} & \phi \\ \phi & \phi & \{c\} \\ \{c\} & \{b\} & \phi \end{pmatrix}$ such that $RA = E$

Step 5 : $PR = \begin{pmatrix} \{a\} & \{c\} & \phi \\ \{c\} & \{b\} & \phi \\ \phi & \phi & \{c\} \end{pmatrix}$ is a g-inverse of A.

It can be verified by

$$A(PR)A= \begin{pmatrix} \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \{c\} \end{pmatrix} \begin{pmatrix} \{a\} & \{c\} & \phi \\ \{c\} & \{b\} & \phi \\ \phi & \phi & \{c\} \end{pmatrix} \begin{pmatrix} \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \{c\} \end{pmatrix} =$$

$$\begin{pmatrix} \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \{c\} \end{pmatrix} = A.$$

Step 6 : Let $Y = (PR)\, A\, (PR)$. Y is a semi inverse of A. it can be verified by $AYA = A$ and $YAY = Y$.

$$AYA = \begin{pmatrix} \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \{c\} \end{pmatrix} \begin{pmatrix} \{a\} & \phi & \phi \\ \phi & \{b\} & \phi \\ \phi & \phi & \{c\} \end{pmatrix} \begin{pmatrix} \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \{c\} \end{pmatrix} =$$

$$\begin{pmatrix} \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \{c\} \end{pmatrix} = A$$

And

$$YAY = \begin{pmatrix} \{a\} & \phi & \phi \\ \phi & \{b\} & \phi \\ \phi & \phi & \{c\} \end{pmatrix} \begin{pmatrix} \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \phi \\ \{a\} & \{b\} & \{c\} \end{pmatrix} \begin{pmatrix} \{a\} & \phi & \phi \\ \phi & \{b\} & \phi \\ \phi & \phi & \{c\} \end{pmatrix} =$$

$$\begin{pmatrix} \{a\} & \phi & \phi \\ \phi & \{b\} & \phi \\ \phi & \phi & \{c\} \end{pmatrix} = Y.$$

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CHAPTER 4

GENERALIZED INVERSES OF INCLINE MATRICES

In [40], it is proved that for a regular element, $a\{1,2\}=a$. This need not be the case for matrices over an incline (refer, Example (3.3.20)). On this basis, in this chapter, we have discussed the existence and construction of reflexive inverses, minimum norm g-inverses, least square g-inverses and Moore - Penrose inverse associated with a matrix over an incline whose idempotent elements are linearly ordered. We provide a formula to compute the Moore - Penrose inverse. We have determined the equivalent conditions for the existence of Moore - Penrose inverse and for an incline matrix to be range symmetric. We discuss the characterization of set of g-inverses and obtain equivalent conditions for the existence of the Moore - Penrose inverse of a range symmetric incline matrix.

4.1. Reflexive Inverses

Lemma 4.1.1

For $A \,\varepsilon\, \pounds_{mn}$, $AA^TA \geq A^{\textcircled{3}}$.

Proof

Let $A = (a_{ij})$ for $i=1$ to m, $j=1$ to n.

Then (il)th element of $AA^TA = (\sum_j(\sum_k a_{ij}\; a_{jk})a_{jl})$. By Incline Property (1.2.1), this expression is greater than (or) equal to each term in the summation. Therefore, for $k = l$, $j=i$ we have

(il)th element of $AA^TA \geq a_{ik}\, a_{ik}\, a_{il}$

$= a_{il}^{3}$

$= (il)$th element of $A \odot A \odot A$

Thus $AA^TA \geq A^{\textcircled{3}}$.

Illustration 4.1.2

Let us consider the incline $£ = \{[0,1], sup\ (x,y), \times\}$, where '$\times$' denotes usual multiplication.

Let $A = \begin{pmatrix} 0.1 & 0.2 \\ 0.3 & 0.5 \end{pmatrix} \varepsilon\ £_2$

$A^T = \begin{pmatrix} 0.1 & 0.3 \\ 0.2 & 0.5 \end{pmatrix} \varepsilon\ £_2$

By the incline operation (+,×) in £, we get

$AA^TA = \begin{pmatrix} 0.03 & 0.05 \\ 0.075 & 0.125 \end{pmatrix}$

$A^{③} = \begin{pmatrix} 0.001 & 0.008 \\ 0.027 & 0.125 \end{pmatrix}$.

Here $AA^TA \geq A^{③}$ and Lemma (4.1.1) holds.

Proposition 4.1.3

Let A_1 *and* $A_2\ \varepsilon\ £_{mn}$ and B_1 and $B_2\ \varepsilon\ £_{np}$. If $A_1 \leq A_2$ and $B_1 \leq B_2$, then $A_1B_1 \leq A_2B_2$.

Proof

Let $A_1 = (a_{ij})$, $A_2 = (a_{ij})$, $B_1 = (b_{jk})$ and $B_2 = (b'_{jk})$. Since $A_1 \leq A_2$ and $B_1 \leq B_2$, we have $a_{ij} \leq a'_{ij}$ and $b_{jk} \leq b'_{jk}$ for all $i=1$ to m, $j=1$ to n, $k=1$ to p. Therefore, $a_{ij}\, b_{jk} \leq a'_{ij} b'_{jk}$. Hence $\sum_j a_{ij}\, b_{jk} \leq \sum_j a'_{ij} b'_{jk}$. Thus $A_1 B_1 \leq A_2 B_2$.

Remark 4.1.4

Both the conditions are essential in Proposition (4.1.3). This is illustrated in the following example:

Example 4.1.5

Let us consider the incline $£$ with 0 and 1 under the operations supremum and usual multiplication.

Let $A_1 = \begin{pmatrix} 0.8 & 0.5 \\ 0.4 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $A_1 \leq A_2$.

Let $B_1 = \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.3 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 0.6 & 0.2 \\ 0.7 & 0.3 \end{pmatrix}$, $B_1 \nleq B_2$.

$$A_1 B_1 = \begin{pmatrix} 0.8 & 0.5 \\ 0.4 & 1 \end{pmatrix} \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.48 & 0.40 \\ 0.4 & 0.3 \end{pmatrix}$$

$$A_2 B_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0.6 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.7 & 0.3 \\ 0.7 & 0.3 \end{pmatrix}$$

Therefore, $A_1 B_1 \nleq A_2 B_2$.

Proposition 4.1.6

Let $A \varepsilon £_{mn}$ and $B \varepsilon £_m$ such that $AA^T \leq B$. Then $A \leq B \odot A$.

Proof

This follows from Proposition (4.1.3) and Lemma (4.1.1).

Theorem 4.1.7

For $A \varepsilon DL_{mn}$, $AA^TA \geq A$.

Proof

Let $A = (a_{ij}) \varepsilon £_{mn}$. Then (it)th element of $AA^TA = (\sum_j (\sum_k a_{ik} a_{jk}) a_{jt})$. By Incline Property (1.2.2), this expression is greater than (or) equal to each term in the summation. Therefore, for $k=t$, $j=i$, we have

$$\begin{aligned} \text{(it)th element of } AA^TA &\geq a_{ik}\, a_{ik}\, a_{it} \\ &= a_{it}^3 \\ &= a_{it} \end{aligned}$$

Thus $AA^TA \geq A$.

Remark 4.1.8

Since by Proposition (1.2.21) and DL = $£$, the Theorem (4.1.7) reduces to the following Corollaries:

Corollary 4.1.9

Let $£$ be a regular incline. If $A \varepsilon £_{mn}$ then $AA^TA \geq A$.

Corollary 4.1.10

Let $£$ be a distributive lattice. If $A \varepsilon £_{mn}$ then $AA^TA \geq A$.

Remark 4.1.11

For matrices over an incline with max-min operation that is, for fuzzy matrices $AA^TA \geq A$ (p.19, [36]).

We observe that in general, Theorem (4.1.7) need not be true for a matrix over an incline that is, $AA^TA \ngeq A$ for $A \ \varepsilon \ \pounds_{mn}$. Thereby, the regularity condition on the incline is essential. This is illustrated in the following example:

Example 4.1.12

Let us consider the incline $\pounds = \{[0,1],+,\cdot\}$ in Illustration (4.1.2).

In this incline, 0 and 1 are the only idempotent elements. Hence by Proposition (1.2.20), £ is not a regular incline.

Let $A = \begin{pmatrix} 0.3 & 0.2 \\ 0.5 & 0.7 \end{pmatrix} \varepsilon \ \pounds_2$.

$A^T = \begin{pmatrix} 0.3 & 0.5 \\ 0.2 & 0.7 \end{pmatrix} \ \varepsilon \ \pounds_2$

By the incline operation $(+,\cdot)$ under the usual multiplication of real numbers,

$$AA^TA = \begin{pmatrix} 0.075 & 0.105 \\ 0.245 & 0.343 \end{pmatrix}$$

Hence $AA^TA \ngeq A$. Thus Theorem (4.1.7), Corollary (4.1.8) and Corollary (4.1.9) fails.

Lemma 4.1.13

Let $A \ \varepsilon \ \pounds_{mn}$. $Y,Z \ \varepsilon \ A\{1\}$ and $X= YAZ$. Then $X \ \varepsilon A\{1,2\}$.

Proof

Since $Y,Z \ \varepsilon A\{1\}$ and $X = YAZ$

$$\begin{aligned} AXA &= A(YAZ)A \\ &= (AYA)ZA && \text{(by Definition (3.1.1))} \\ &= AZA. && \text{(by Definition (3.1.1))} \end{aligned}$$

and $XAX = (YAZ)A(YAZ)$

$$= \quad Y(AZA)\,(YAZ)$$

$$= \quad Y\,(AXA)\,(YAZ)$$

$$= \quad YA\,(YAZ) \qquad \text{(by Definition (3.1.1))}$$

$$= \quad Y(AYA)Z$$

$$= \quad YAZ \qquad \text{(by Definition (3.1.1))}$$

$$= \quad X.$$

Hence $X \varepsilon A\,\{1,2\}$.

Theorem 4.1.14

Let £ be an incline whose idempotent elements are linearly ordered. A is a regular matrix and $X \varepsilon A\,\{1\}$, $X \varepsilon A\,\{2\}$ if and only if $\mathcal{R}(AX) = \mathcal{R}(X)$.

Proof

Since A is regular and $X \varepsilon A\{1\}$, AX is regular being idempotent. By Proposition (3.2.2), $\mathcal{R}(A) = \mathcal{R}(AX)$

$$X \;\varepsilon A\,\{2\} \quad \Rightarrow XAX = X$$

$$\Rightarrow A \,\varepsilon\, X\,\{1\}$$

$$\Rightarrow \mathcal{R}(X) = \mathcal{R}(AX)$$

Conversely, Let $\mathcal{R}(X) = \mathcal{R}(AX)$. By Lemma (1.2.8),

$\mathcal{R}(X) \subseteq \mathcal{R}(AX) \Rightarrow X = YAX$ for some $Y \varepsilon £_m$.

$$X(AX) \;= (YAX)\,AX$$

$$XAX \;= Y(AXA)X$$

$$= YAX$$

$$XAX \;= X.$$

Thus $X \varepsilon A\,\{2\}$.

Corollary 4.1.15

Let £ be an incline whose idempotent elements are linearly ordered. For $A \;\varepsilon\; D£_{mn}$, if $X \varepsilon A\,\{1,2\}$ then $\rho(A) = \rho(X)$.

Proof

Since $X \varepsilon A \{1\}$, A is regular, AX is regular being idempotent. Therefore by Proposition (3.2.2), $\rho(A) = \rho(AX)$. Since $X \varepsilon A \{1,2\}$, by Theorem (3.1.4), $\mathcal{R}(X) = \mathcal{R}(AX)$. Therefore $\rho(A) = \rho(AX) = \rho(X)$.

Remark 4.1.16

It is well known that for complex matrices (p.19,[5]) any two of the statement $X \varepsilon A \{1\}$, $X \varepsilon A \{2\}$ and rank X = rank A implies the other. However this fails for fuzzy matrices and for matrices over an incline.

$X \varepsilon A \{1\}$ and $\rho(X) = \rho(A)$ need not imply $X \varepsilon A \{2\}$. This is illustrated in the following example:

Example 4.1.17

Let us consider the incline $£ = \{[0,1], sup\ (x,y), min(x,y)\}$.

For $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0.5 \end{pmatrix} \varepsilon A\{1\}$ and $\rho(X) = \rho(A) = 2$.

But $XAX = \begin{pmatrix} 0 & 1 \\ 1 & 0.5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0.5 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \neq X$.

Hence $X \notin A \{2\}$.

Remark 4.1.18

In Theorem (4.1.14), the condition $X \varepsilon A \{1\}$ is essential. This is illustrated in the following example:

Example 4.1.19

Let us consider the incline £ in Example (4.1.17).

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$AXA = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \neq A \Rightarrow X \notin A \{1\}$.

$AX = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $\mathcal{R}(X) = \mathcal{R}(AX)$.

But $XAX = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $XAX \neq X$. Hence $X \notin A \{2\}$ and Theorem (4.1.14) fails.

4. 2 . Minimum Norm g-inverses, Least square g-inverses and Moore - Penrose Inverse

In this section, we discuss the existence and construction of minimum norm (or) *{1,3}* inverses, least square (or) *{1,4}* inverses and Moore - Penrose inverse of a matrix over an incline.

Theorem 4.2.1

For $A \ \varepsilon \ \pounds_{mn}$, A has a *{1,3}* inverse $\Leftrightarrow$ A^TA is a regular matrix and $\mathcal{R}(A^TA) = \mathcal{R}(A)$.

Proof

Let X be a *{1,3}* inverse of A. Then $AXA = A$ and $(AX)^T = AX$. Therefore, $A^TA = A^TAXA$ and $A=AXA = (AX)^TA = X^TA^TA$. Then by Lemma (1.2.8), $\mathcal{R}(A) \subseteq \mathcal{R}(A^TA)$ and $\mathcal{R}(A^TA) \subseteq \mathcal{R}(A)$. Therefore $\mathcal{R}(A) = \mathcal{R}(A^TA)$. Since A has a {1,3} inverse, automatically implies that A has a {1} inverse. Then, A is regular together with $\mathcal{R}(A) = \mathcal{R}(A^TA)$ implies A^TA is regular, by Theorem (3.2.5).

Conversely, Let A^TA be a regular matrix and $\mathcal{R}(A) = \mathcal{R}(A^TA)$. Then A is a regular matrix follows from Theorem (3.2.5). Since A^TA is regular, define $Y = (A^TA)^-A^T$ for some g-inverse of A^TA.

By Lemma (1.2.8), $\mathcal{R}(A) \subseteq \mathcal{R}(A^TA) \Rightarrow A=XA^TA$ for some $X \varepsilon \ \pounds_m$ $\rightarrow$(4.2.1)

$$
\begin{aligned}
AY &= XA^TA\,(A^TA)^-A^T \\
&= XA^TA\,(A^TA)^-A^TAX^T \\
&= X(A^TA)(A^TA)^-(A^TA)X^T \\
&= XA^TAX^T \qquad \text{(By Definition (3.1.1))} \\
&= XA^T.
\end{aligned}
$$

Now post multiplying by A and using (4.2.1), we get $AYA = XA^TA=A$. Hence $Y \ \varepsilon \ A\ \{1\}$.

$(AY)^T =(XA^T)^T = (AX^T) = XA^TAX^T = XA^T = AY.$

Therefore, $Y \varepsilon \ A\ \{3\}$. Thus A has a *{1,3}* inverse.

Theorem4.2.2

For $A \varepsilon £_{mn}$, A has a $\{1,4\}$ inverse $\Leftrightarrow AA^T$ is regular and $\mathcal{C}(AA^T) = \mathcal{C}(A)$.

Proof

A has a $\{1,4\}$ inverse $\Leftrightarrow A^T$ has A $\{1,3\}$ inverse (by Remark (1.2.7))
$\Leftrightarrow AA^T$ is regular and $\mathcal{R}(A^T) = \mathcal{R}(AA^T)$ (by Theorem (4.2.1))
$\Leftrightarrow AA^T$ is regular and $\mathcal{C}(A) = \mathcal{C}(AA^T)$ (by Theorem (4.2.1)).

Lemma 4.2.3

Let $£$ be an incline whose idempotent elements are linearly ordered. $A = (a_{ij}) \varepsilon DL_m$ is symmetric idempotent incline matrix $\Leftrightarrow a_{ij} A_{i*} = A_{i*} A_{j*}$.

Proof

We have $a_{ij} A_{i*} = a_{ji} A_{i*} \leq A_{j*}$ and $a_{ij}A_{i*} \leq A_{i*}$. So $a_{ij} A_{i*} \leq A_{i*} A_{j*}$. Suppose for some k, $a_{ik}a_{jk} > a_{ij}a_{ik}$. Then $a_{ik}\ a_{jk} > a_{ij}$ .Yet $a_{ij} \leq a_{ik}a_{kj} = a_{ik}a_{jk}$. This contradicts that $a_{ij}A_{i*} = A_{i*}A_{j*}$.

Conversely, let $v\langle i\rangle$ for $i=1$ to r be a set of independent incline vectors such that $v\langle i\rangle v\langle j\rangle = x\, v\langle i\rangle = x\, v\langle j\rangle$ for some x, for each i,j. Let $A \varepsilon DL_m$ such that $a_{ij} = \sup_k \{ v\langle k\rangle_i\, v\langle k\rangle_j \}$.

We will show that A is symmetric idempotent having the span of the $v\langle i\rangle$ as its row space. Let $v\langle m\rangle_u$ be a maximal entry in $v\langle m\rangle$. Suppose

$v\langle j\rangle_u \geq v\langle m\rangle_u$ for some $j \neq m$.

Then let x be such that

$v\langle m\rangle\, v\langle j\rangle = x\, v\langle m\rangle = x\, v\langle j\rangle$

Then $x \geq v\langle m\rangle_u\, v\langle j\rangle_u = v\langle m\rangle_u$

Therefore $x\, v\langle m\rangle = v\langle m\rangle$

Therefore $x\, v\langle j\rangle = v\langle m\rangle$

Therefore $v\langle m\rangle$ is dependent. This is a contradiction.

So $v\langle m\rangle_u > v\langle j\rangle_u$ for all $j \neq m$.

We will next show that $A_{u*} = v\langle m\rangle$. We have $a_{ui} = sup_k \ \{v\langle k\rangle_u \ v\langle k\rangle_i\} \geq v\langle m\rangle_u \ \langle m\rangle_i = v\langle m\rangle_i$. Therefore $A_{u*} \geq v\langle m\rangle$. Suppose $a_{ui} > v\langle m\rangle_i$ for some i. Then for some k, $v\langle k\rangle_u \ v\langle k\rangle_i > v\langle m\rangle_i$. So $v\langle k\rangle_u > v\langle m\rangle_i$ *and* $v\langle k\rangle_i > v\langle m\rangle_i$. So $v\langle m\rangle_i \ v\langle k\rangle_i = v\langle m\rangle_i$. *So* $v\langle m\rangle_i$ cannot be a maximal entry of $v\langle m\rangle$, so $v(m)$, $v\langle m\rangle_u > v\langle m\rangle_i$. Let x be such that $v\langle m\rangle \ v\langle k\rangle = x \ v\langle m\rangle = x \ v\langle k\rangle$. Then $x \geq v\langle m\rangle_u \ v\langle k\rangle_u = v\langle k\rangle_u$ *and* $v\langle m\rangle_i \ v\langle k\rangle_i = x \ v\langle k\rangle_i$.

Therefore $v\langle m\rangle_i = x \ v\langle k\rangle_i$. *Yet* $x \ v\langle k\rangle_i \geq v\langle k\rangle_i \ v\langle k\rangle_u > v\langle m\rangle_i$. This is contradiction. Therefore $A_{u*} = v\langle m\rangle$. This proves $v\langle i\rangle$ appears as a row of A, for each i.

Therefore the row space of A contains the span of the vectors $v\langle i\rangle$. Yet also we have $A_{i*} = \sum_k v\langle k\rangle_i \ v\langle k\rangle$ so the row space of A is contained in the span of the vectors $v\langle i\rangle$. So the two subspaces are equal.

It follows from the definition of A that A is symmetric. It remains to show that A is idempotent.

By Theorem (2.3.13) it suffices to prove that any zero pattern of A is idempotent. But on any zero pattern, $v\langle i\rangle \, v\langle j\rangle = x \ v\langle i\rangle = x \ v\langle j\rangle$ means $\bar{v}\langle i\rangle = \bar{v}\langle j\rangle = 1$ *(or)* $\bar{v}\langle i\rangle = \bar{v}\langle j\rangle = 0$ according as x goes to 1 or 0 in the zero pattern. Let $\tilde{A}$ denote the image of A in the zero pattern. Then if $\tilde{a}_{ij} = 1$ and $\tilde{a}_{jk} = 1$ then for some m, $\tilde{v}\langle m\rangle_i = 1$ and $\tilde{v}\langle m\rangle_j = 1$ and for some h, $\tilde{v}\langle h\rangle_j = 1$ and $\tilde{v}\langle h\rangle_k = 1$. Therefore $\tilde{v}\langle m\rangle = \tilde{v}\langle h\rangle$, since $\tilde{v}\langle m\rangle = \tilde{v}\langle h\rangle \neq 0$. Therefore $\tilde{v}\langle m\rangle_k = 1$. Therefore $\tilde{a}_{ik} = 1$. This proves $\tilde{A}$ is transitive, ie., $\tilde{A}^2 \leq \tilde{A}$. But a transitive and symmetric Boolean matrix is idempotent, since $\tilde{A}^3 \geq \tilde{A}$ for $\tilde{A}$ symmetric. So each zero pattern $\tilde{A}$ is idempotent. This completes the proof.

Theorem 4.2.4

Let $£$ be an incline whose idempotent elements are linearly ordered. For $A \ \varepsilon \ DL_{mn}$, the following statements are equivalent:

(i) A has a $\{1,3\}$ inverse

(ii) For any two row basis vectors A_{i*} and A_{j*}

$A_{i*}A_{j*} = xA_{i*} = xA_{j*}$ for some $x \ \varepsilon \ £$.

Proof

A has a *{1,3}* inverse if and only if there exists a symmetric idempotent matrix X, such that $\mathcal{R}(X) = \mathcal{R}(A)$ that is, if G is a *{1,3}* inverse then take $X = GA$. Conversely, if X exists, choose G so that $GA = X$. Then G will be a *{1,3}* inverse of A. Thus to prove the theorem, it is enough to prove the condition $A_{i*}A_{j*} = xA_{i*} = xA_{j*}$ for basis vectors that characterizes the row spaces of symmetric idempotent incline matrices. Since G is symmetric idempotent, this follows from Lemma (4.2.3).

Theorem 4.2.5

Let £ be an incline whose idempotent elements are linearly ordered. For $A \varepsilon DL_{mn}$, the following statements are equivalent:

(i) A has a *{1,4}* inverse

(ii) For any two column basis vectors A_{*i}, A_{*j},

$$A_{*i}\, A_{*j} = xA_{*i} = xA_{*j} \text{ for } x \,\varepsilon\, £.$$

Proof

This can be proved in the same manner as that of Theorem (4.2.4) and hence omitted.

Theorem 4.2.6

Let $A \,\varepsilon\, £_{mn}$ be a regular matrix, with $\mathcal{C}(A^T) = \mathcal{C}(A^TA)$, then A^TA is a regular matrix and $Y = (A^TA)^-A^T \,\varepsilon\, A\{1,2,3\}$, for some $(A^TA)^- \,\varepsilon\, (A^TA)\{1\}$.

Proof

Since $A \,\varepsilon\, £_{mn}$ is a regular matrix, with $\mathcal{C}(A^T) = \mathcal{C}(A^TA)$ then by Theorem (3.2.5) A^TA is a regular matrix. $\mathcal{C}(A^T) \subseteq \mathcal{C}(A^TA)$ implies $A^T AX = A^T$, for some $X \,\varepsilon\, £_{nm}$. Taking transpose on both sides we get,

$$\begin{aligned} A &= X^TA^TA \\ AYA &= (X^TA^TA)\,((A^TA)^-A^T)A \\ &= X^T(A^TA)\,(A^TA)^-\,(A^TA) \end{aligned}$$

$$
\begin{aligned}
&= X^T A^T A && \text{(By Definition (3.1.1))}\\
&= A.\\
YAY &= Y\,(X^T A^T A)\,(A^T A)^- A^T\\
&= YX^T(A^T A)(A^T A)^-\,(A^T AX)\\
&= YX^T(A^T A)\,(A^T A)^-(A^T A)X\\
&= YX^T A^T AX && \text{(By Definition (3.1.1))}\\
&= YAX\\
&= ((A^T A)^- A^T)AX\\
&= (A^T A)^-\,(A^T AX)\\
&= (A^T A)^-(A^T)\\
&= Y\\
AY &= (X^T A^T A)(A^T A)^- A^T\\
&= X^T(A^T A)(A^T A)^-(A^T AX)\\
&= X^T(A^T A)\,(A^T A)^-\,(A^T A)\\
&= X^T A^T AX && \text{(By Definition (3.1.1))}\\
&= AX\\
(AY)^T &= (AX)^T\\
&= X^T A^T\\
&= X^T A^T AX\\
&= AX\\
&= AY
\end{aligned}
$$

Thus $Y \varepsilon A\ \{1,2,3\}$.Hence the Theorem.

Theorem 4.2.7

Let $A\ \varepsilon\ £_{mn}$ be a regular matrix with $\mathcal{R}(A^T) = \mathcal{R}(A\,A^T)$ then AA^T is a regular matrix and $Z = A^T(A\,A^T)^-\ \varepsilon\ A\ \{1,2,4\}$, for some $(AA^T)^-\ \varepsilon\ (AA^T)\ \{1\}$.

Proof

This can be proved along the same lines as that of Theorem (4.2.6) and hence omitted.

Theorem 4.2.8

Let $A \varepsilon £_{mn}$ be a regular matrix. Then $A^{(1,4)}A\,A^{(1,3)} = A^{\dagger}$.

Proof

$$
\begin{array}{lcll}
\text{Let } X & = & A^{(1,4)}A\,A^{(1,3)}, \text{ by Lemma (4.1.13) } X \varepsilon A\,\{1,2\}. & \\
AX & = & A\,(A^{(1,4)}A\,A^{(1,3)}) & \\
 & = & (A\,A^{(1,4)}A)\,A^{(1,3)}) & \\
 & = & A\,A^{(1,3)} & \text{(by Definition (3.1.1))} \\
(AX)^T & = & (A\,A^{(1,3)})^T & \\
 & = & A\,A^{(1,3)} & \text{(by Definition (1.2.6))} \\
 & = & AX & \\
 & = & (A^{(1,4)}\,A\,A^{(1,3)})\,A & \\
 & = & A^{(1,4)}(A\,A^{(1,3)}A) & \\
 & = & A^{(1,4)}A & \text{(by Definition (3.1.1))} \\
(XA)^T & = & (A^{(1,4)}\,A)^T & \\
 & = & A^{(1,4)}\,A & \text{(by Remark (1.2.7))} \\
 & = & XA & \\
\text{Thus } X & = & A^{(1,4)}\,A\,A^{(1,3)} = A^{\dagger}. \text{ Hence the theorem.} &
\end{array}
$$

Lemma 4.2.9

Let $A \varepsilon £_{mn}$ be a regular matrix. $A\,A^TA = A \Leftrightarrow A^{\dagger}$exists and $A^{\dagger} = A^{T.}$

Proof

Since $A\,A^TA = A;\ A^T$ is a g-inverse of A. The existence of $A^{\dagger}$, directly follows from the fact that A^T satisfies the defining equations in Definition (1.2.6) of the Moore Penrose inverse $A^{\dagger}$. Converse is trivial.

Theorem 4.2.10

Let $£$ be an incline whose idempotent elements are all linearly ordered. For $A \varepsilon DL_{mn}$, the following statements are equivalent:

(i) AA^T and A^TA are regular, $\mathcal{R}(A^T) = \mathcal{R}(A\,A^T)$ and $\mathcal{C}(A^T) = \mathcal{C}(A^TA)$

(ii) A has a *{1,3}* inverse and *{1,4}* inverse

(iii) For any two row basis vectors A_{i*} and A_{j*} and any two column basis vectors A_{*i} and A_{*j}, $A_{i*} A_{j*} = xA_{i*} = xA_{j*}$ and $A_{*i} A_{*j} = yA_{*i} = yA_{*j}$ for $x, y \,\varepsilon\, £$

(iv) A^T is a g- inverse of A

(v) Each zero pattern of A has a Moore - Penrose inverse

(vi) $A^{\dagger}$exists and equals A^T.

Proof

(i) ⇔ (ii) : This follows directly from Theorem (4.2.1)and (4.2.2).

(ii) ⇔ (iii) : This follows from Theorems (4.2.4) and (4.2.5).

(ii) ⇔ (vi) : (ii) ⇒ (vi) follows from Theorem(4.2.8) and converse is trivial.

(iv) ⇔ (vi) : This equivalence is precisely Lemma (4.2.9

(iv) ⇔ (v) : If each zero pattern A_{α} of A has a Moore - Penrose inverse, then by Lemma (4.2.9) $A_{\alpha}^{\dagger} = A_{\alpha}^{T}$. By Theorem (2.3.18)

$A^T = \left(\sum_{\alpha\varepsilon\phi_A} \alpha A_{\alpha}\right)^T = \sum_{\alpha\varepsilon\phi_A} \alpha A_{\alpha}^T = \sum_{\alpha\varepsilon\phi_A} \alpha\, A_{\alpha}^{\dagger} = A^{\dagger}$

Thus, $A^T = A^{\dagger}$ is the Moore - Penrose inverse of A and hence A^T is a g-inverse of A.

Remark 4.2.11

By Lemma (2.3.13), for $\alpha,\beta \,\varepsilon\, £$, $\alpha \leq \beta \Leftrightarrow \alpha + \beta = \beta \Leftrightarrow \alpha\, \beta = \alpha$, which is precisely the max-min composition in £. Hence the Theorems (4.2.4), (4.2.5) and (4.2.10) valid for a regular incline whose elements are all linearly ordered. Hence we shall state the following corollaries without proofs.

Corollary 4.2.12

Let $£$ be a regular incline whose elements are all linearly ordered and $A \,\varepsilon\, £_{mn}$, then the following statements are equivalent:

(i) A has a $\{1,3\}$ inverse

(ii) For any two row basis vectors A_{i*} and A_{j*}.

$A_{i*}A_{j*} = xA_{i*} = xA_{j*}$ for some $x \varepsilon £$.

Corollary 4.2.13

Let $£$ be a regular incline whose elements are all linearly ordered and $A \varepsilon £_{mn}$, then the following statements are equivalent:

(i) A has a $\{1,4\}$ inverse

(ii) For any two column basis vectors A_{*i}, A_{*j},

$A_{*i}\, A_{*j} = xA_{*i} = xA_{*j}$ for $x \varepsilon £$.

Corollary 4.2.14

Let $£$ be a regular incline whose elements are all linearly ordered and $A \varepsilon £_{mn}$, then the following statements are equivalent:

(i) AA^T and A^TA are regular, $\mathcal{R}(A^T) = \mathcal{R}(AA^T)$ and $\mathcal{C}(A^T) = \mathcal{C}(A^TA)$

(ii) A has a $\{1,3\}$ inverse and $\{1,4\}$ inverse

(iii) For any two row basis vectors A_{i*} and A_{j*} any two column basis vectors A_{*i} and A_{*j}. $A_{i*}\, A_{j*} = xA_{i*} = xA_{j*}$ and $A_{*i}\, A_{*j} = yA_{*i} = yA_{*j}$ for x, $y \varepsilon £$.

(iv) A^T is a g- inverse of A.

(v) Each zero pattern of A has a Moore - Penrose inverse.

(vi) $A^{\dagger}$ exists and equals A^T.

Remark 4.2.15

Since $DL = £$, by Proposition (1.2.21) for a regular incline, the above Corollaries remain valid for a distributive lattice whose elements are all linearly ordered.

Next we shall discuss the equivalent conditions for the existence of the Moore- Penrose inverse of a range symmetric incline matrix.

Theorem 4.2.16

Let $£$ be an incline whose idempotent elements are linearly ordered.

For $A \varepsilon DL_n$, the following statements are equivalent:

(i) A is range symmetric and there is a basis $\{v_1,v_2,....v_r\}$ of $\mathcal{R}(A)$ such that for any two basis vectors v_i an v_j there is an element $x \varepsilon £$ satisfying $v_iv_j = xv_i = xv_j$.

(ii) $A^\dagger$ exists and $A^\dagger = A^T$ is range symmetric.

(iii) There exists a symmetric idempotent incline matrix E such that $AE = EA$ and $\mathcal{R}(A) = \mathcal{R}(E)$.

Proof

(i) $\Rightarrow$ (ii) Since A is range symmetric, $\mathcal{R}(A) = \mathcal{R}(A^T) = \mathcal{C}(A) \Rightarrow$ A and A^T are regular $\Rightarrow A$ has a $\{1,3\}$ and $\{1,4\}$ inverse and by Theorem (4.2.1) and Theorem (4.2.2), $\mathcal{R}(A)$ and $\mathcal{C}(A)$ satisfy the given condition that $v_iv_j = xv_i = xv_j$ for some $x \varepsilon £ \Rightarrow A^\dagger$ exists and $A^\dagger = A^T$ $\mathcal{R}((A^T))^T = \mathcal{R}(A^T)$. Since A is range symmetric $\mathcal{R}((A^T))^T = \mathcal{R}(A^\dagger)$. Thus $A^\dagger$ is range symmetric.

(ii) $\Rightarrow$ (iii) Since $A^\dagger$exists, $A^\dagger = A^T$ and $E=A^\dagger A$ is symmetric idempotent incline matrix. $AE = AA^\dagger A = A \Rightarrow \mathcal{R}(A) \subseteq \mathcal{R}(E)$ and $A^\dagger A = E \Rightarrow \mathcal{R}(E) \subseteq \mathcal{R}(A)$. Hence $\mathcal{R}(A) = \mathcal{R}(E)$. Since $A^\dagger = A^T$ is range symmetric, $\mathcal{R}(A^T) = \mathcal{R}(A) = \mathcal{R}(E)$. By Lemma (1.2.10), $\mathcal{R}(A^T) \subseteq \mathcal{R}(E) \Rightarrow A^T = A^T E^- E$, since E is symmetric idempotent.

$$A^T = A^T E$$
$$(A^T)^T = (A^T E)^T$$
$$A = E^T A = EA.$$

Thus $E = A^\dagger A$ is symmetric idempotent incline matrix, $AE = EA$, $\mathcal{R}(A) = \mathcal{R}(E)$.

(iii) => (i) Since E is symmetric idempotent, $E^{\dagger} = E$ exists. By Theorem (4.2.10), there is a basis $\{v_1, v_2 .,....v_i\}$ of $\mathcal{R}(E) = \mathcal{R}(A)$ such that for any two basis vectors v_i and v_j, $v_iv_j = xv_i = xv_j$ for $x \,\varepsilon\, \pounds$. By Theorem (4.2.10), A has a $\{1,4\}$ inverse G and $GA = EE^{\dagger} = E$. Hence $AE = A(GA) = A$. Since $AE = EA = A$, taking the transpose $EA^T = A^TE = A^T$ By Lemma (1.2.9), $A^TE = A^T \Rightarrow \mathcal{R}(A^T) \subseteq \mathcal{R}(E) = \mathcal{R}(A)$ again by Lemma (1.2.9), $EA^T = A^T \Rightarrow \mathcal{R}(A^T) \Rightarrow \mathcal{R}(A^T) \subseteq \mathcal{R}(E)$. Hence $\mathcal{R}(A) = \mathcal{R}(A^T)$. Thus (i) holds.

In particular for a regular incline whose elements are all linearly ordered Theorem (4.2.16) reduce to the following Corollaries:

Corollary 4.2.17

Let $\pounds$ be a regular incline whose elements are all linearly ordered. If E is symmetric idempotent incline matrix and $A \,\varepsilon\, \pounds_n$ such that $\mathcal{R}(A) = \mathcal{R}(E)$ then the following are equivalent :

(i) A is range symmetric

(ii) $AE = EA$

(iii) $A^{\dagger}$exists and range symmetric.

Corollary 4.2.18

Let $\pounds$ be a regular incline whose elements are all linearly ordered. Let $A \,\varepsilon\, \pounds_n$.If $A^{\dagger}$exists, then the following are equivalent:

(i) A is range symmetric

(ii) $AA^{\dagger} = A^{\dagger}A$

(iii) $A^{\dagger}$is range symmetric

(iv) A is normal.

Example 4.2.19

Let us consider the incline $\pounds = \{[0,1], sup\ (x,y), inf(xy)\}$

For $A = \begin{pmatrix} 0.5 & 0 \\ 0.3 & 0 \end{pmatrix}$, $A^{\dagger} = A^T$ exists.

$A^2 = A$, $\rho(A^2) = \rho(A)$. $\mathcal{R}(A) \neq \mathcal{R}(A^T)$, A is not range symmetric.

4.3. Characterizations of set of g - inverses

In this section, we shall discuss the characterization of set of all g-inverses of an incline matrix. We shall determine equivalent conditions for various g-inverses of a range symmetric incline matrix to be range symmetric and generalized inverses belonging to the sets $A\{1,2\}$, $A\{1,2,3\}$ and $A\{1,2,4\}$ of a range symmetric matrix A are characterized.

Theorem 4.3.1

Let $A \varepsilon £_{mn}$, $B \varepsilon £_{pq}$ be regular matrices and $D \varepsilon £_{mq}$. Then the incline matrix equation $A X B = D$ is solvable if and only if $AA^{-}DB^{-}B = D$ for $A^{-} \varepsilon A\{1\}$ and $B^{-} \varepsilon B\{1\}$.

Proof

Let $D = AA^{-}DB^{-}B$

$= A(A^{-}DB^{-})B$

Hence $X = A^{-}DB^{-}$ is a solution of $A X B = D$. Thus $A X B = D$ is solvable.

Conversely, let X be any solution $A X B = D$.

Then $D = AA^{-}A X B B^{-}B$

$= AA^{-}(A X B) B^{-}B$

$D = AA^{-}DB^{-}B$

Hence the Theorem

Theorem 4.3.2

Let $xA = b$ be a incline relational equation, $b \varepsilon £_{1n}$, $A \varepsilon £_{mn}$. If X satisfies $A X A = A$ then $x = bX$ is a solution of $xA = b$.

Proof

In Theorem (4.3.1). Put $A = I$, $B = A$, $D = b$, then $xA = b$ is solvable if and only if $bA^{-}A = b$

$$\begin{aligned} xA &= (bX)A \\ &= (bA^{-}AX)A \\ &= bA^{-}(AXA) \end{aligned}$$

$$= \quad bA^{-}A$$

$$xA \quad = \quad b$$

Thus $x = b\,X$ is a solution of $xA=b$.

Corollary 4.3.3

If $X \varepsilon A\,\{1\}$ then $xA=b$ has a solution if and only if $(bX)\,A=b$.

Proof

Let $xA=b$ has a solution.

Since $x=bX$ and $b = bA^{-}A$.

$$\begin{aligned}(bX)A &= (bA^{-}AX)A \\ &= bA^{-}(AXA) \\ &= bA^{-}A \\ &= b\end{aligned}$$

Conversely, if $(bX)A=b$ then $x=bX$ is a solution of $xA=b$.

We derive the characterization of the set $A\{1\}$ in terms of a particular element of the set.

Lemma 4.3.4

For $A \varepsilon \pounds_{mn}$ if G^{*} and G are g-inverses of A such that $G^{*} \geq G$, then $G+H$ is a g-inverse of A for some $H \varepsilon \pounds_{nm}$ such that $G^{*} \geq G+H \geq G$.

Proof

Let $G^{*}- G =H$. Then $G^{*} \geq H$

Since $G^{*} \geq G$ and $G^{*} \geq H$, it follows that $G^{*} \geq G+H \geq G$.

Then $AG^{*}A \geq A(G+H)A \geq AGA$

$\Rightarrow \quad A \geq A(G+H)\,A \geq A.$

$\Rightarrow \quad A(G+H)\,A=A.$

Thus $(G+H)$ is a g-inverse of A.

Theorem 4.3.5

Let A ε $\pounds_{mn}$ and G be a particular g-inverse of A. Then $A_G\{1\}$ $= \{G+H/$ for all incline matrix $H \varepsilon \pounds_{nm}$ such that $A \geq AHA\}$ $\rightarrow$ (4.3.1) is the set of all g-inverses of A dominating G.

Proof

Let $\mathcal{B}$ denote the set on the RHS of (4.3.1). Suppose $G^* \varepsilon A_G \{1\}$, then $G^* \geq G$. Let $G^*- G = H$.

By Lemma (4.3.4), $G^* \geq G+H \geq G$ and $G+H$ is a g-inverse of A dominating G.

Further, $A\,(G+H)A=A \Rightarrow AGA + AHA = A$

$\Rightarrow A+AHA = A$

$\Rightarrow A \geq AHA.$

Hence $G+H \varepsilon \mathcal{B}$. Thus for each $G^* \varepsilon A_G \{1\}$, there exists a unique element in $\mathcal{B}$. Conversely, for any $G^* \varepsilon \mathcal{B}$.

$G^* = G+H \geq G,$ with $A \geq AHA$

$$\begin{aligned} \text{Now, } AG^*A &= A(G+H)A \\ &= AGA + AHA \\ &= A+AHA \\ &= A \end{aligned}$$

Thus $G^* \varepsilon A_G \{1\}$.

Corollary 4.3.6

For $A \varepsilon \pounds_n$ be an idempotent incline matrix. Then $\{G+H/$ for all incline matrix $H \varepsilon \pounds_n$ such that $A \geq AHA\}$ is the set of all A dominating A.

Proof

This follows from Theorem (4.3.2), by taking $G=A$. Since A is a idempotent matrix, A itself is a g-inverse.

Next we discuss the characterization of the sets $A\{1,3\}$ and $A\ \{1,4\}$ interms of a particular element of the set. The key to the characterization of the set $A\ \{1,3\}$ is the following :

Theorem 4.3.7

The set $A\ \{1,3\}$ consists of all solutions for X of $AX = AG$, where G is a $\{1,3\}$ inverse of A.

Proof

Since $G\ \varepsilon\ A\ \{1,3\}$, by Definition (1.2.6), $AGA = A$ and $(AG)^T = AG$.

For $X\ \varepsilon\ A\ \{1,3\}$ we have $AXA = A$ and $(AX)^T = AX$

$$
\begin{aligned}
\text{Then} \qquad AG &= (A\,X\,A)G \\
&= (AX)\,(AG) \\
&= (AX)^T(AG)^T \\
&= (X^TA^T)\,(G^TA^T) \\
&= X^T\,(A^TG^TA^T) \\
&= X^TA^T \\
&= (AX)^T \\
&= AX
\end{aligned}
$$

Hence X is a solution of $AX = AG$.

Conversely, let $AG = AX$ with $G\ \varepsilon\ A\ \{1,3\}$.

$$
\begin{aligned}
\text{Then} \qquad & A = AGA \\
& \Rightarrow A = AXA \\
& \Rightarrow X\ \varepsilon\ A\ \{1\}
\end{aligned}
$$

Since $AX = AG$

$$
\begin{aligned}
& \Rightarrow (AX)^T = (AG)^T \\
& \Rightarrow (AX)^T = AG \\
& \Rightarrow (AX)^T = AX \\
& \Rightarrow X\ \ \varepsilon\ A\ \{3\}
\end{aligned}
$$

Thus $X\ \varepsilon\ A\ \{1,3\}$.

Theorem 4.3.8

For $A \varepsilon £_{mn}$ and $G \varepsilon A\{1,3\}$, $A_G\{1,3\} = \{G+H/$ for all incline matrices $H \varepsilon £_{nm}$ such that $AG \geq AH\}$ $\rightarrow$ (4.3.2)

is the set of all $\{1,3\}$ inverses of A dominating G.

Proof

Let $\mathcal{B}$ denote the set on the RHS of (4.3.2). Suppose $G^* \varepsilon A_G\{1,3\}$, then $G^* \geq G$. Let G^*-$G = H$.

Since $A_G\{1,3\} \subseteq A_G\{1\}$, by Lemma (4.3.4), $G^* \geq G+H \geq G$

$\Rightarrow$ $AG^* = A(G+H) \geq AG$.

By Theorem (4.3.7), $G^* \varepsilon A_G\{1,3\}$ and $G \varepsilon A_G\{1,3\}$,

$\Rightarrow$ $AG^* = AG$

$\Rightarrow$ $A(G+H) = AG$

$\Rightarrow$ $AG \geq AH$.

Hence $G+H \varepsilon \mathcal{B}$. Thus for each $G^* \varepsilon A_G\{1,3\}$, there exists an unique element in $\mathcal{B}$. Conversely, for any $G^* \varepsilon \mathcal{B}$, $G^* = G+H \geq G$ with $AG \geq AH$. Hence $AG^* = AG + AH = AG$. By Theorem (4.3.7), it follows that $G^* \varepsilon A_G\{1,3\}$.

Corollary 4.3.9

Let $A \varepsilon £_n$ be a symmetric and idempotent incline matrix. Then $\{A+H/$ for all incline matrix $H \varepsilon £_n$ such that $AG \geq AH\}$ is the set of all $\{1,3\}$ inverses of A dominating A.

Proof

This follows from Theorem (4.3.8) by taking $G=A$. Since A is symmetric and idempotent incline matrix, A itself is a $\{1,3\}$ inverse.

Remark 4.3.10

The condition that G is a $\{1,3\}$ inverse of A is essential in Theorem (4.3.7) and Theorem (4.3.8). This is illustrated in the following example:

Example 4.3.11

Consider the incline $£ = \{[0,1], sup\ (x,y), inf\ (x,y)\}$.

For $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \varepsilon\ A\ \{1,3\} \Rightarrow A\ \{1,3\} \neq \phi$.

Consider $G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \varepsilon\ A\ \{1,3\}$.

$$AG = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

For $H = \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix}$,

$$AH = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix}$$

$$\Rightarrow AG \geq AH.$$

But $G+H = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \quad G+H \notin A\ \{3\}$

$\Rightarrow \quad G+H \notin A_G\ \{1,3\}$. Since $G\ \varepsilon\ A\ \{1,3\} \Leftrightarrow G^T\ \varepsilon\ A^T\ \{1,4\}$

Theorem 4.3.12

The set $A\ \{1,4\}$ consists of all solutions for X of $XA = GA$, where G is a $\{1,4\}$ inverse of A.

Proof

This can be proved in the same manner as that of Theorem (4.3.7).

Theorem 4.3.13

For $A\ \varepsilon\ £_{mn}$ and $G\ \varepsilon\ A\ \{1,4\}$

$A_G\ \{1,4\} = \{G+H/$ for all incline matrix $H\ \varepsilon\ £_{nm}$ such that $GA \geq HA\} \rightarrow$ (4.3.3)

is the set of all $\{1,4\}$ inverse of A dominating G.

Proof

Let $\mathcal{B}$ denote the set on the RHS of (4.3.3). Suppose $G^*\ \varepsilon\ A_G\ \{1,4\}$, then $G^* \geq G$. *Let* $G^*\text{-}G = H$. Since $A_G\ \{1,4\} \subseteq A_G\{1\}$, by Theorem (4.2.7), $G^* \geq G+H \geq G \Rightarrow G^*A \geq (G+H)\ A \geq GA$.

By Theorem (4.3.12), $G^* \varepsilon A_G\{1,4\}$ and $G \varepsilon A_G\{1,4\} \Rightarrow G^*A = GA$

$\Rightarrow (G+H)A = GA$

$\Rightarrow GA \geq HA$

Thus $G+H \varepsilon \mathcal{B}$. Hence for each $G^* \varepsilon A_G \{1,4\}$, there exists an unique element in $\mathcal{B}$. Conversely, for any $G^* \varepsilon B$, $G^* = G+H \geq G$ with $GA \geq HA$. Hence $G^*A = GA + HA = GA$. By Theorem (4.3.12), $G^* \varepsilon A_G \{1,4\}$.

Corollary 4.3.14

Let $A \varepsilon £_n$ be a symmetric and idempotent incline matrix. Then $\{A+H/$ for all incline matrix $H \varepsilon £_n$ such that $GA \geq HA\}$ is the set of all $\{1,4\}$ inverses of A dominating A.

Proof

This follows from Theorem (4.3.13) by taking $G=A$. Since A is a symmetric idempotent incline matrix, A itself is a $\{1,4\}$ inverse.

Remark 4.3.15

In Theorem (4.3.12), G is a $\{1,4\}$ inverse of A is essential. This is illustrated in the following example:

Example 4.3.16

Consider the incline £ in Example (4.3.11).

For $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \varepsilon A\{1,4\} \Rightarrow A\{1,4\} \neq \phi$.

Consider $G = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \notin A\{1,4\}$

$GA = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

Consider $H = \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix}$

$HA = \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0.1 & 0.1 \end{pmatrix} \Rightarrow GA \geq HA.$

But $G+H = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \Rightarrow G+H \notin A\{4\}$

$\Rightarrow G+H \notin A_G\{1,4\}$.

Theorem 4.3.17

Let A be symmetric idempotent matrix in $\pounds_n$ then $A^{\dagger}=A$.

Proof

By Theorem (4.3.8) and Theorem (4.3.13),

$A^{(1,3)} = A+K$ where $A \geq AK$ and $A^{(1,4)} = A+H$ where $A \geq HA$. $\rightarrow (4.3.4)$

By Proposition (1.2.14) $A+AK = A = A+HA$. $\rightarrow (4.3.5)$

By Theorem (4.2.8), for some $A^{(1,3)}$ and $A^{(1,4)}$ inverses of A,

$$\begin{aligned}
A^{\dagger} &= A^{(1,4)}A\,A^{(1,3)} \\
&= (A+H)\,A\,(A+K) && \text{by (4.3.4)} \\
&= (A^2 + HA)\,(A+K) \\
&= (A + HA)\,(A + K) \\
&= A\,(A+K) && \text{by (4.3.5)} \\
&= A^2 + AK \\
&= A + AK \\
A^{\dagger} &= A && \text{by (4.3.5)}
\end{aligned}$$

Hence the Theorem.

Remark 4.3.18

We have proved that, A is range symmetric $\Leftrightarrow A^{\dagger}$ is range symmetric in Corollary (4.2.18). Thus the range symmetric property of an incline matrix is preserved for its Moore - Penrose inverse. However, all other g-inverses of range symmetric matrix need not be range symmetric.

Now, we show that any $X \in A\{1,2\}$, $X \in A\{1,2,3\}$ and $X \in A\{1,2,4\}$ of a range symmetric matrix A is also range symmetric under certain conditions in the following Theorems.

Theorem 4.3.19

Let $A \in \pounds_n$, $X \in A\{1,2\}$ and AX, XA are range symmetric. Then A is range symmetric if and only if X is range symmetric.

Proof

Since $X \varepsilon A\ \{1,2\}$, by Definition (1.2.6), $AXA = A$, $XAX = X$, AX and XA are range symmetric, by Definition (2.5.1), $\mathcal{R}(AX) = \mathcal{R}((AX)^T)$ and $\mathcal{R}(XA) = \mathcal{R}((XA)^T.)$ By Proposition (3.2.2), $\mathcal{R}(A) = \mathcal{R}(XA) = \mathcal{R}((XA)^T) = \mathcal{R}(A^TX^T) = \mathcal{R}(X)^T$. Again by Proposition (3.2.2), $\mathcal{R}(A^T) = \mathcal{R}(X^TA^T) = \mathcal{R}(AX)^T = \mathcal{R}(AX) = \mathcal{R}(X)$.

A is range symmetric $\Leftrightarrow \mathcal{R}(A) = \mathcal{R}(A^T)$

$\Leftrightarrow \mathcal{R}(X^T) = \mathcal{R}(X)$

$\Leftrightarrow$ X is range symmetric.

Theorem 4.3.20

Let $A\ \varepsilon\ £_n$, $X\ \varepsilon\ \{1,2,3\}$, $\mathcal{R}(A) = \mathcal{R}(X^T)$. Then A is range symmetric if and only if X is range symmetric.

Proof

Since $X \varepsilon A\{1,2,3\}$, by Definition (1.2.6), $AXA = A$, $XAX = X$, $(AX)^T = AX$

Now, $\mathcal{R}(A^T) = \mathcal{R}(X^TA^T) = \mathcal{R}(AX)^T = \mathcal{R}(AX) = \mathcal{R}(X)$

A is range symmetric $\Leftrightarrow \mathcal{R}(A) = \mathcal{R}(A^T)$

$\Leftrightarrow \mathcal{R}(X^T) = \mathcal{R}(X)$

$\Leftrightarrow$ X is range symmetric.

Theorem 4.3.21

Let $A\ \varepsilon\ £_n$, $X\ \varepsilon\ A\ \{1,2,4\}$, $\mathcal{R}(A^T) = \mathcal{R}(X)$. Then A is range symmetric if and only if X is range symmetric.

Proof

Since $X\ \varepsilon\ \{1,2,4\}$, by Definition (1.2.6), $AXA =A$, $XAX = X$, $(XA)^T = XA$.

Now, $\mathcal{R}(A) = \mathcal{R}(XA)= \mathcal{R}(XA)^T = \mathcal{R}(A^TX^T) = \mathcal{R}(X^T)$

A is range symmetric $\Leftrightarrow \mathcal{R}(A) = \mathcal{R}(A^T)$

$\Leftrightarrow \mathcal{R}(X^T) = \mathcal{R}(X)$

$\Leftrightarrow$ X is range symmetric.

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CHAPTER 5

INCLINE RELATIONAL EQUATIONS AND APPLICATIONS

We have discussed the consistency of incline relational equations, that is, equations of the form $xA=b,$ where A is a matrix and b is a vector over an incline £ with least element 0 and greatest element 1.We have determined the existence of the maximum solution of $xA=b$ under the condition that each column of A is comparable with the corresponding components of the vector b in £. This includes the result found in [42] as a special case for fuzzy relational equations. By using the maximum solution of the incline relational equation, we have discussed when a vector can be expressed uniquely as a linear combination of the standard basis vectors. As a special case, we have exhibited that each vector in a regular incline whose elements are all linearly ordered has a unique decomposition as a linear combination of its standard basis vectors, which we call as standard incline linear combination. This is a generalization of standard linear combination of a vector over the max-min Fuzzy algebra [37]. We have highlighted an application of incline matrices in Automata Theory. We have discussed the structure of DFSA and NDFSA. We obtained that any set can be represented as an incline, which is a DFSA and the conversion of a NDFSA to an equivalent incline, which is a DFSA by way of constructing an incline structure with the set of all states in the given NDFSA. In this way, we exhibited that the corresponding transition matrices are space equivalent. We have applied the matrices over an incline £ under the operations addition as supermum and usual multiplication in Cryptography for encryption and decryption based on McEliece scheme [3].We obtain the equivalence of finite state machines in Automata. v

5.1. Incline Relational Equations

In this section, we discuss the consistency of the equation of the form

$$x\,A = b \qquad \rightarrow(5.1.1)$$

with $x = [x_j / j \varepsilon N_m]$, $b = [b_k / k\varepsilon\, N_n]$ and $A= (a_{ij})\ \varepsilon\ £_{mn}$; where N_r denotes the set of all positive integers 1 to r. Let $\Omega\ (A,b)$ denotes the set of all solutions of (5.1.1).

Definition 5.1.1

If $\Omega\ (A,b)$ is a comparable set then any element $\hat{x}$ of $\Omega\ (A,b)$ is called a maximal solution of $x\,A=b$ if for all $x\varepsilon\ \Omega\ (a,b)$, $x \geq \hat{x}$ implies $x= \hat{x}$.

Definition 5.1.2

If $\Omega\ (A,b)$ is a comparable set then any element $\check{x}$ *of* $\Omega\ (A,b)$ is called a minimal solution of $x\,A=b$ if for all $x\ \varepsilon\ \Omega\ (A,b)$, $x \leq \check{x}$ implies $x = \check{x}$.

Lemma 5.1.3

Let $xA = b$ be as in equation (5.1.1). If $\sum_j (a_{jk}) < b_k$ for some $k\ \varepsilon\ N_n$, then $\Omega\ (A,b) = \phi$.

Proof

If $\sum_j (a_{jk}) < b_k$, then by using Incline Properties (1.2.2) and (1.2.1), we have

$$x_j a_{jk} \leq\ a_{jk} \leq \textstyle\sum_j (a_{jk})$$

$$\textstyle\sum_j (x_j a_{jk})\ \leq\ \sum_j (a_{jk}) < b_k$$

Hence no values of $x_j\ \varepsilon$ £ satisfies the equation $x\,A=b$. Therefore $\Omega\ (A,b) = \phi$.

Remark 5.1.4

The condition $\sum_j (a_{jk}) < b_k$ in Lemma (5.1.3) is essential. This is illustrated in the following:

Illustration 5.1.5

Consider the incline £ = *([0,1], sup (x,y), ×)*, '×' denotes the usual multiplication.

Let us consider the equation $x\,A=b$,

where $A=\begin{pmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{pmatrix} \varepsilon\, £_2$ and $b=(0.9 \quad 0.1)$ $\varepsilon\, £^2$ are given.

Since $0.6 < 0.9$, the condition $\sum_j (a_{jk}) < b_k$ holds for the 1st column and $0.7 \not< 0.1,$ the condition $\sum_j (a_{jk}) < b_k$ fails for the 2nd column.

Hence $\Omega\,(A,b) = \phi$.

Theorem 5.1.6

Let $xA = b$ be as in (5.1.1), such that A_{*k}, the kth column of A and b_k, the kth component of b are comparable for each k. Then $\Omega\,(A,b) \neq \phi$ if and only if $\hat{x} = [\hat{x}_j \,/ j\,\varepsilon\, N_m]$ defined as

$$\hat{x}_j = \min \sigma\,(a_{jk}, b_k) \qquad \rightarrow (5.1.2)$$

Where $\sigma\,(a_{jk}, b_k) = \begin{Bmatrix} b_k & \text{if} & a_{jk} > b_k \\ 1 & \text{otherwise} & \end{Bmatrix}$

is the maximum solution.

Proof

If $\Omega\,(A,b) \neq \phi$ then $\hat{x}$ is a solution of equation (5.1.2). For if $\hat{x}$ is not a solution, then $\hat{x}\,A \neq b$. Hence $\sum_j \hat{x}_j\, a_{jk0} \neq b_{ko}$ for atleast one $k_o\, \varepsilon N_n$. By definition of $\hat{x}$, since $\hat{x}_j \leq b_k$ for each k, $\hat{x}_j \leq b_{ko}$. By our assumption, $\sum_j (a_{jk0}) < b_{ko}$ for some $k_o\, \varepsilon N_n$ and by Lemma (5.1.3), $\Omega\,(A,b) = \phi$ which is a contradiction. Hence $\hat{x}$ is a solution of equation (5.1.1). Next, let us prove that $\hat{x}$ is the maximum solution. If possible let us assume that y be a solution of equation (5.1.1) such that $y > \hat{x}$, then by using Definition (1.2.13), $y_j > \hat{x}_j$ for each $j\, \varepsilon\, N_m$, that is $y_{jo} > \hat{x}_{jo}$ for $j_o\, \varepsilon\, N_m$. Since A_{*k} is comparable with b_k, for each $k\, \varepsilon\, N_n$, $\sigma\,(a_{jk}, b_k)$ can be determined. Therefore, by Definition of $\hat{x}$, we have

$y_{jo} > \hat{x}_{j0} = min\ \sigma\ (a_{jk,}b_k)$. Since $\Omega\ (A,b) \neq \phi$ by Lemma (5.1.3), $\sum_j(a_{jk}) > b_k$ for each $k\ \varepsilon\ N_n$. Hence $b_{ko} \neq \Sigma(y_j a_{jko})$, for $k_0\ \varepsilon\ N_n$ which contradicts our assumption $y\ \varepsilon\ \Omega\ (A,b)$. Therefore $\hat{x}$ is the maximum solution of equation (5.1.2). Converse is trivial.

Remark 5.1.7

In the above Theorem (5.1.6), the condition that the *k*th column of *A* and the *k*th component of *b* to be comparable is essential in determining the maximum solution. This is illustrated in the following example:

Example 5.1.8

Let us consider the incline $£ = \{0,a,b,c,d,1\}$ in Example (2.1.8).

Let us consider the equation $xA=b$, where $A=\begin{pmatrix} 1 & c \\ b & 0 \end{pmatrix}\ \varepsilon £_2$ and b = $[d\ 0]\ \varepsilon\, £^2$ are given.

Since $1 > d$, the condition $\sum_j(a_{jk}) < b_k$ in Lemma (5.1.3) fails for the *1*st column. Since $c > 0$, the condition $\sum_j(a_{jk}) < b_k$ in Lemma (5.1.3) fails for the 2nd column. Therefore, $\Omega\ (A,b) \neq \phi$.

Next to determine the solution set $\Omega\ (A,b)$.

$$[x_1\ x_2]\begin{pmatrix} 1 & c \\ b & 0 \end{pmatrix} = [d\ 0]$$

$x_1 1 + x_2\, b = d$ and $x_1\, c = 0$. Since $x_1\, c = 0$, for every

$x_1 \varepsilon\ \{0, a\}$ and $x_1 1 +\ x_2\, b = d,$ for every $x_1,\ x_2 \varepsilon\ \{1, b, c, d\}$.

Therefore $\Omega\ (A,b)\ = \{\ (0,1)\ (0,b)\ (0,c),\ (0,d),\ (a,1)\ (a,b)\ (a,c)\ (a,d)\}$.

In $\Omega\ (A,b)$, the elements (a,b) and (a,c) are not comparable. Therefore $\Omega\ (A,b)$ is not a comparable set. Hence by Definition (5.1.1), it has no maximum element. Thus $xA=b$ has no maximum solution. However, $\Omega\ (A,b) \neq \phi$.

Thus Theorem (5.1.6) fails.

Remark 5.1.9

If the elements of £ are linearly ordered then the comparability of A_{*k} and b_k for $k \varepsilon N_n$ automatically holds. Hence Theorem (5.1.6) reduces to the following:

Corollary 5.1.10

Let £ be an incline whose elements are all linearly ordered and the equation $xA=b$ be as in (5.1.1). Then $\Omega\ (A,b) \neq \phi$ if and only if $\hat{x}$ defined as in (5.1.2) is the maximum solution.

Theorem 5.1.11

Let £ be an incline whose idempotent elements are all linearly ordered and the equation $xA=b$ *with* $x=[x_j\ /j\ \varepsilon\ N_m],\ b=[b_k/k\varepsilon N_n]\ \varepsilon DL^n$ and $A\ \varepsilon\ DL_{mn}$. Then $\Omega\ (A,b) \neq \phi$ if and only if $\hat{x}$ defined as in (5.1.2), is the maximum solution.

Proof

Since idempotent elements are all linearly ordered, for $A\ \varepsilon\ DL_{mn}$, $b\ \varepsilon\ DL^n$, the comparability of A_{*k} and b_k automatically holds for each $k\ \varepsilon\ N_n$. Then the theorem can be proved in a similar manner as that of Theorem (5.1.6).

Corollary 5.1.12

Let £ be a regular incline whose elements are all linearly ordered and the equation $xA=b$ be as in (5.1.1). Then $\Omega\ (A,b) \neq \phi$ if and only if $\hat{x}$ defined as in (5.1.2), is the maximum solution.

Proof

Since £ is a regular incline by Proposition (1.2.20) each element of £ is idempotent and therefore *DL=£.* Then the rest follows from Theorem (5.1.11).

Remark 5.1.13

Since by Proposition (1.2.21), a commutative regular incline is a distributive lattice and *DL=£,* Theorem (5.1.11) reduces to the following:

Corollary 5.1.14

Let £ be a distributive lattice whose elements are all linearly ordered and the equation $xA=b$ be as in (5.1.1). Then $\Omega\ (A,b) \neq \phi$ if and only if $\hat{x}$ defined as in (5.1.2), is the maximum solution.

Remark 5.1.15

The condition that $A\ \varepsilon\ DL_{mn}$ in Theorem (5.1.11) is essential. The condition that the elements are to be linearly ordered cannot be relaxed in the above Corollaries (5.1.10), (5.1.12) and (5.1.14). These are illustrated in the following example:

Example 5.1.16

Let us consider the incline £ in Example (2.1.8), whose idempotent elements 0 and d are linearly ordered.

For the equation $xA=b,$

where $A= \begin{pmatrix} 1 & c \\ d & 0 \end{pmatrix} \notin\ DL_2$ and $b = [d\ 0]\ \varepsilon\ DL^2$.

In Example (5.1.8), we have seen that $\Omega\ (A,b) \neq \phi$ but $xA=b$ has no maximum solution.

Thus Theorem (5.1.11) fails.

Example 5.1.17

Let us consider the incline £ $= (\mathcal{P}(D),\ \cup,\ \cap),$ in Example (2.3.16).Here, £ is a commutative regular incline hence by Proposition (1.2.21), £ is a distributive lattice whose elements are all idempotent but not linearly ordered. For instance, $\{a\}$, $\{b\}$ and $\{c\}$ are not comparable.

Let us consider the equation $x\,A=b,$

where $A= \begin{pmatrix} \{a,c\} & \{c\} \\ \{a\} & \{a,b,c\} \end{pmatrix}\ \varepsilon$ $£_2$ and $b = [\{c\}\ \{b,c\}]\ \varepsilon£^2$ are given.

Here the condition $\sum_j (a_{jk}) < b_k$ fails for both the columns. Therefore by Lemma (5.1.3), $\Omega\ (A,b) \neq \phi$.

Next to determine the solution set Ω *(A,b)*

$$[x_1\ x_2]\ \begin{pmatrix} \{a,c\} & \{c\} \\ \{a\} & \{a,b,c\} \end{pmatrix} = [\{c\}\ \{b,c\}]$$

On computation we get,

$x_1\ \varepsilon\ \{\{c\}\ \{b,c\}\}$ and $x_2\ \varepsilon\ \{\{b\}\ \{b,c\}\}$

Therefore $\Omega(A,b)=\{(\{c\},\{b\})\ (\{c\},\{b,c\})\ (\{b,c\},\{b\})\ (\{b,c\},\{b,c\})\}$.

In Ω *(A,b),* the elements *({c}, {b,c})* and *({b,c}, {b})* are not comparable. Hence by Definition (5.1.1), $xA=b$ has no maximum solution. However $\Omega\ (A,b) \neq \phi$.

Thus Corollaries (5.1.10), (5.1,12) and (5.1.14) fail.

Remarks 5.1.18

It is well known that (p.2, [1]), every Fuzzy algebra is an incline. However, the converse not true. This can be seen from the incline

$£ = \{[0,1], sup(x,y), \times\}$ where '×' is the ordinary multiplication. £ is an incline whose elements are all linearly ordered. Here, for any $x,y\ \varepsilon £$ if $x \leq y$ then $x+y$ = y but $xy \neq x$. Therefore £ is not the max-min fuzzy algebra.

If the equation (5.1.1) is a fuzzy relational equation that is, *A* is a matrix over the max-min fuzzy algebra then Theorem (5.1.6) reduces to the following result of Sanchez [42] (p.70, [36]).

Corollary 5.1.19

Let $xA=b$ be the fuzzy relational equation as in (5.1.1). Then $\Omega(A,b) \neq \phi$ if and only if $\hat{x}$ defined as in (5.1..2), is the maximum solution.

Next we shall describe a method for finding the minimum solutions of incline relational equation (5.1.1). The following method enables us to generalize the method provided in [23] by Higashi and Klir (Quoted in [36], p.70).

First we have to find the maximum solution of an incline relational equation $xA=b$, for which $\Omega\ (A,b)\neq\phi$ and the kth column of A is comparable with the kth component of b. The following method for finding the minimum solution is possible only if the components of b' are comparable.

Let b' be the vector whose components are that of b in the equation $xA=b$ arranged such that $b'_1 > b'_2 > \ldots > b'_n$. Then $b' = b\,Q$ for some permutation matrix Q. Let $A' = A\ Q$. Then the equation $xA=b$ and $xA' = b'$ will have the same set of solutions. That is, $\Omega\ (A,b) = \Omega\ (A',b')$. Hence let us describe a method for determining all minimal solutions of equation (5.3.1) based on the assumption that the components of b in (5.3.1) are ordered such that $b'_1 > b'_2 > \ldots > b'_n$. Further, let us take that $\Omega\ (A,b) \neq \phi$ and $\hat{x}$ be the maximum solution. We note that the equation (5.3.1) reduced to simpler forms in the following cases:

Case 1

In the maximum solution $\hat{x}$, if any component $\hat{x}_j=0$, then $x_j=0$ for all $x\ \varepsilon\ \Omega\ (A,b)$. Further $x_j\, a_{ji} = 0$ for each $i\ \varepsilon\ N_n$ and therefore $x_jA_{j*}=0$. So we can eliminate $\hat{x}_j$ from $\hat{x}$ and also the jth row A_{j*} from the matrix A in equation (5.3.1).

Case 2

If $b_k=0$ for some $k\ \varepsilon\ N_n$ in the vector b of (5.3.1) then $\hat{x}A_{*k} = b_k=0$, for each $x\ \varepsilon\ \Omega\ (A,b)$, $x<\hat{x}$, $x\,A_{*k} = \hat{x}A_{*k} = b_k = 0$.

In such cases arise, then after performing these eliminations we obtain solutions of the reduced equation. The solution of the original equation is obtained by simply inserting zeros at the locations that were eliminated in the reduction step. Now, let us consider the equation $xA=b$ of the form (5.3.1) with $\Omega\ (A,b) \neq \phi$, $b_k \neq 0$ for all $k\ \varepsilon\ N_n$ and all the components of its maximum solution $\hat{x}$ are non zero. The set $\breve{\Omega}\ (A,b)$ of all minimum solution of the reduced equation $xA=$b can be determined by the following procedure:

Step 1:

Determine the sets $J_k(\hat{x}) = \{j \varepsilon N_m / min\left(\hat{x}_j, a_{jk}\right) = b_k\}$ for all $k \varepsilon N_n$. Construct their Cartesian product $J(\hat{x}) = J_1(\hat{x}) \times J_2(\hat{x}) \times \ldots\ldots \times J_n(\hat{x})$.

Step 2:

Denote the elements of $J(\hat{x})$, by $\beta = (\beta_k / k\varepsilon N_n)$. For each $\beta \varepsilon J(\hat{x})$ and each $j \varepsilon N_m$, determine the set $k(\beta,j) = \{k \varepsilon N_m \ \beta_k = j\}$.

Step 3:

For each $\beta \varepsilon J(\hat{x})$ generate the m- tuple $g(\beta) = g_j\{\beta) j\varepsilon N_m\}$

Where $g_j(\beta) = \begin{cases} max_{k \varepsilon k(\beta,j)} b_k & \text{if } k(\beta,j) \neq \phi \\ 0 & \text{Otherwise.} \end{cases}$

Step 4:

From all the m – tuples $g(\beta)$ generated in step 3, select only the minimal ones by pair wise comparison. The resulting set of m-tuples is the set $\Omega\ (A,b)$ of all minimal solution of the reduced form of equation (5.3.1).

Step 5:

Extend all minimal solutions of the reduced equation of (5.3.1) by zeros at the locations that were eliminated in the reduction step, this results in the solution set $\Omega\ (A,b)$ of equation (5.3.1).

Illustration 5.1.20

Let us consider the incline $£ = \{[0,1], max\ \{x,y\}, min\{x,y\}\}$

Given $A = \begin{pmatrix} 0.9 & 0.5 & 1 \\ 0.8 & 0.7 & 0.4 \\ 0.5 & 0.3 & 0.6 \end{pmatrix}$ and $b = (0.7 \quad 0.6 \quad 0.5)$

First to check $\Omega\ (A,b) \neq \phi$, by using Lemma (5.1.3)

Let us consider the equation $xA = b$,

where $A = \begin{pmatrix} 0.9 & 0.5 & 1 \\ 0.8 & 0.7 & 0.4 \\ 0.5 & 0.3 & 0.6 \end{pmatrix}$ and $b = (0.7 \quad 0.6 \quad 0.5)$

Since $0.9 > 0.7$, the condition $\sum a_{jk} < b_k$ fails for the 1st column. Since $0.7 > 0.6$, and $0.6 > 0.5$, the condition fails for the 2nd and 3rd column.

Hence $\Omega\ (A,b) \neq \phi$.

Next to find the maximum solution $\hat{x}$:

From Theorem (5.1.6), we get

$\hat{x}_1 = \min \sigma(a_{jk}, b_k) = \min \sigma\{(a_{11},b_1)\ (a_{12},b_2)\ (a_{13},b_3)\}$

$= \min \sigma\{0.7,1,0.5\} = 0.5$

$\hat{x}_2 = \min \sigma(a_{2k}, b_k) = \min \sigma\{(a_{21},b_1)\ (a_{22}, b_2)\ (a_{23},b_3)\}$

$= \min \sigma\{0.7,0.6,1\} = 0.6$

$\hat{x}_3 = \min \sigma(a_{3k}, b_k) = \min \sigma\{(a_{31}..b_1)\ (a_{32}\ldots b_2)\ (a_{33}\ldots b_3)\}$

$= \min \sigma\{1,1,0.5\} = 0.5$

Therefore, $\hat{x}\ A=b$

$$(0.5 \ \ 0.6 \ \ 0.5)\begin{pmatrix} 0.9 & 0.5 & 1 \\ 0.8 & 0.7 & 0.4 \\ 0.5 & 0.3 & 0.6 \end{pmatrix} = (0.7 \ \ 0.6 \ \ 0.5)$$

$\hat{x} = (0.5 \ \ 0.6 \ \ 0.5)$ is the maximum solution of $xA = b$.

Next to determine minimal solutions:

Step 1 : To determine $J_k(\hat{x})$ for $k = 1,2,3$

$J_k(\hat{x}) = \{j \varepsilon N_3 / \min(\hat{x}_j, a_{jk})\} = b_k$

$J_1(\hat{x}) = \{j \varepsilon N_3/\min (\hat{x}_j, a_{j1}))\ \} = b_1$

$= \{j \varepsilon N_3/\min \{(\hat{x}_1,a_{11})(\ \hat{x}_2,a_{21})\ (\hat{x}_3,a_{31})\} = b_1\}$

$= \{j \varepsilon N_3/\min \{(0.5,\ 0.9)\ (0.6,0.8)\ (0.5,0.5)\} = 0.7\ \}$

$= \{1\}$

$J_2(\hat{x}) = \{j \varepsilon N_3/\min (\hat{x}_j, a_{j2}) = b_2\}$

$= \{j \varepsilon N_3/\min \{(\hat{x}_1,a_{12})\ (\hat{x}_2.\ a_{22})\ (\hat{x}_3,a_{32})\} = b_2\}$

$= \{j \varepsilon N_3/\min \{(0.5,0.5)\ (0.6,0.7)\ (0.5,0.3) = 0.6\}$

$= \{2\}$

$$J_3(\hat{x}) = \{j \varepsilon N_3 / \min(\hat{x}_j, a_{j3}) = b_3\}$$

$$= \{j \varepsilon N_3 / \min\{(\hat{x}_1, a_{13})\ (\hat{x}_2,\ a_{23})\ (\hat{x}_3,\ a_{33})\} = b_3\}$$

$$= \{j \varepsilon N_3 / \min\{(0.5,1)\ (0.6,0.4)\ (0.5,\ 0.6)|\} = 0.5\}$$

$$= \{1,3\}$$

$$J(\hat{x}) = J_1(\hat{x}) \times J_2(\hat{x}) \times J_3(\hat{x})$$

$$= \{0\} \times \{2\} \times \{1,3\}$$

$$= \{(0,2,1),\ (0,2,3)\ (2,1,3)\}$$

Step 2:

To determine the sets $K(\beta,j)$ for each $\beta = J(\hat{x})$ and for each $j \varepsilon N_j$

For $\beta = (0,2,1)$,

$$K(\beta,1) = \{k \varepsilon N_3 / \beta_k = 1\} = \{3\}$$
$$K(\beta,2) = \{k \varepsilon N_3 / \beta_k = 2\} = \{2\}$$
$$K(\beta,3) = \{k \varepsilon N_3 / \beta_k = 3\} = \phi$$

For $\beta = (0,2,3)$,

$K(\beta,1) = \phi$, $K(\beta,2) = \{2\}$ and $K(\beta,3) = \{3\}$

For $\beta = (2,1,3)$,

$K(\beta,1) = \{2\}$, $K(\beta,2) = \{1\}$ and $K(\beta,3) = \{3\}$

Thus the sets $K(\beta,j)$ for each $\beta \varepsilon J(\hat{x})$ and $j \varepsilon N_3$ are listed in the following table:

K(β,j)	***1***	***2***	***3***
$\beta = (0,2,1)$	$\{3\}$	$\{2\}$	ϕ
$\beta = (0,2,3)$	ϕ	$\{2\}$	$\{3\}$
$\beta = (2,1,3)$	$\{2\}$	$\{1\}$	$\{3\}$

Step 3:

For each $\beta \varepsilon J(\hat{x})$ we generate the triples $g(\beta)$

For $\beta = (0,2,1)$,

$$g_1(\beta) = \begin{cases} \max\limits_{k \varepsilon K(\beta,1)} b_k & \text{if } K(\beta,1) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

$= \quad max\ \{b_1, b_2\} = max\ \{0.7, 0.6\} = 0.7$

$$g_2(\beta) = \begin{cases} \max\limits_{k \varepsilon K(\beta,2)} b_k & if\ \ k(\beta,2) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

Since K$(\beta,2) = \{2\}$, $g_2(\beta) = 0.6$

Since $K(\beta,3) = \phi$, $g_3(\beta) = 0$

Thus for $\beta = (0,2,1)$ the triple $g(\beta) = (0.7,0.6,0)$

For $\beta = (0,2,3)$,

Since $K(\beta,1) = \phi$, $g_1(\beta) = 0$

Since K $(\beta,2) = \{2\}$, $g_2(\beta) = 0.6$

Since $K(\beta,3) = \{3\}$, $g_3\{\beta\} = 0.5$

For $\beta = (2,1,3)$

Since $K(\beta,1) = \{2\}$, $g_1(\beta) = 0.6$

Since $K(\beta,2) = \{1\}$, $g_2(\beta) = 0.7$

Since $K(\beta,3) = \{3\}$, $g_3\{\beta\} = 0.5$

β	$g(\beta)$
$(0,2,1)$	$(0.7,0.6,0)$
$(0,2,3)$	$(0,0.6,0.5)$
$(2,1,3)$	$(0.6,0.7,0.5)$

Step 4:

By pairwise comparison we observe that the triples (0,0.6,0.5) is the minimal solution of the equation.

Step 5:

Thus $\widehat{\Omega}(A,b)$ = (0,0.6,0.5). The complete set of solutions $\Omega\ (A,b)$ are the maximum solution $\hat{x}$ = (0.5, 0.6, 0.5) and the minimal solution $\hat{x}$ = (0,0.6,0.5), and the set $\{x \varepsilon P | \hat{x} \leq x \leq \hat{x}\}$.

5.2. Standard incline linear combinations

In this section, we consider a finite subspace S of $£^n$ which has a standard basis. We discuss when a vector in $£^n$ a can be expressed uniquely as a linear combination of its standard basis vectors, which we call it as standard incline linear combination, we determine the standard incline linear combination of the basis vectors.

Theorem 5.2.1

Let $£$ be an incline, S be a finitely generated subspace of $£^n$ and let $\{c_1, c_2, \ldots\ldots c_n\}$ be the standard basis for S. For $x = (x_1, x_2, \ldots x_i, \ldots . x_n) \varepsilon S$ and $c_j = \{c_{j1}, c_{j2}, \ldots c_{jn}, \ldots c_{jk}\}$ if c_{jk} and x_k are comparable for each $j=1$ to n and each k. Then x can be expressed uniquely as a linear combination of the standard basis vectors.

Proof

Since $\{c_1, c_2, \ldots c_n\}$ is a standard basis for S, x is a linear combination of the standard basis vectors.

Let $x = \sum_{j=1}^{n} \beta_j c_j$ where $\beta_j \varepsilon £$.

In this expression, the coefficients β_j*'s* are not unique. If we write this in the matrix form as $x = (\beta_1, \beta_2, \ldots\ldots, \beta_n)\, C$, where C is the matrix whose rows are the basis vectors, then $x = p.C$ has a solution, that is, $\Omega\,(c,x) \neq \phi$. Since the kth column of C_j, for $j=1$ to n is comparable with the kth component of x; by using Theorem (5.1.1), it follows that this equation has a unique maximum solution $\hat{x} = (p_1, p_2, \ldots . p_n)$ (Say). Then $x = \sum_{j=1}^{n} p_j c_j$ where $p_j \varepsilon £$ is the unique representation of the vector x. we call this representation as the standard linear combination of the vector $x \varepsilon £^m$.

Remark 5.2.2

In the above Theorem (5.2.1), the condition that the kth column of C whose rows are the basis vectors and the kth component of x to be comparable is essential. This is illustrated in the following example:

Example 5.2.3

Let us consider the set incline $\pounds = (\mathcal{P}(D), \cup, \cap)$, where $\mathcal{P}(D)$ is the power set of D with set inclusion "$\subseteq$" as the order relation "$\leq$" in Example (2.3.16).

Here, $\pounds$ is a commutative regular incline hence by Proposition (1.2.21), $\pounds$ is a distributive lattice whose elements are all idempotent; but elements are not comparable for instance, $\{a\}, \{b\}$ and $\{c\}$ are not comparable.

Consider the set $S = (\{a\}, \{b\}, \{c\})$. In S each element cannot be expressed as a linear combination of the remaining elements. Hence S is the smallest linearly independent set that is, $<S> = \pounds$. Every element of $\pounds$ can be expressed as a linear combination of the basis vectors. For $\{a\} \varepsilon \pounds$, α is any subset of D containing $\{a\}$, β is any subset of D not containing $\{b\}$ and γ is any subset of D not containing $\{c\}$. Thus $\{a\} = \alpha \{a\}$ for $\alpha \varepsilon \pounds$. Hence S is unique standard basis for $\pounds$.

Any element $x \neq D \varepsilon \pounds$ can be expressed as $x = \alpha\{a\} + \beta\{b\} + \gamma\{c\}$ for suitable choice of $\alpha, \beta, \gamma \varepsilon \pounds$. However, this expression is not unique in the sense that the coefficients of α, β, γ are not uniquely determined by x, since the kth component $x = (\{a\}\ \{b\}\ \{c\})$ is not comparable with the

kth column of $C = \begin{pmatrix} \{a\} \\ \{b\} \\ \{c\} \end{pmatrix}$.

Thus Theorem (5.2.1) fails.

In this incline, $x = D = \{a,b,c\}$ is the only element which is comparable with C. So D is the only element of £ that has the standard linear combination. We can find the standard linear combination of the element $D \varepsilon \pounds$, by determining the maximum solution of the incline relational equation $yC = x$, by using Theorem (5.1.6)

The maximum solution $\hat{y} = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$ is determined by

$\hat{y}_j = min_{k \varepsilon K} \ \sigma(c_{jk}, x_k)$. Here,

$$\hat{y}_1 = \sigma(c_{11}, x) = 1$$
$$\hat{y}_2 = \sigma(c_{21}, x) = 1$$
$$\hat{y}_3 = \sigma(c_{31}, x) = 1$$

Therefore, $\hat{y} = (1,1,1) = (D,D,D)$ Here, D is the largest element of the incline. Thus $D = D\{a\} + D\{b\} + D\{c\}$ is the standard Linear combination of the element $D \varepsilon$ £.

Corollary 5.2.4

Let £ be a regular incline whose elements are all linearly ordered, S be a finitely generated subspace of £^{n} and let $\{c_1, c_2, \ldots. c_n\}$ be the standard basis for S. For $x \varepsilon S$, then x has a standard linear combination.

Proof

Since £ is a regular incline by Proposition (1.2.20) each element of £ is idempotent and therefore DL = £. Then the rest follows from Theorem(5.2.1).

Corollary 5.2.5

Let £ be a distributive lattice whose elements are all linearly ordered, S be a finitely generated subspace of £^{n} and let $\{c_1, c_2, \ldots.. c_n\}$ be the standard basis for S. For $x \varepsilon S,$ then x has a standard linear combination.

Proof

Since by Proposition (1.2.21), a commutative regular incline is a distributive lattice and therefore DL = £. This can be proved in the same manner as that of Theorem (5.2.1) and hence omitted.

Remark 5.2.6

For regular incline, whose elements are all linearly ordered then the comparability condition in Theorem (5.2.1) automatically holds. Hence, the above Theorem (5.2.1) reduces to the following corollary found in [28] (Quoted in [5], p.10)

Corollary5.2.7

Let $\mathcal{F}$ be a Fuzzy algebra, S be a finitely generated subspace of V^n and let $\{c_1,c_2,\ldots..c_n\}$ be the standard basis for S. Then any vector $x \varepsilon S$ can be expressed uniquely as a linear combination of the standard basis vectors.

Next we will see the standard incline linear combination of the basis vectors.

Theorem 5.2.8

Let £ be an incline whose idempotent elements are all linearly ordered and $\{c_1,c_2,\ldots.c_n\}$ be the standard basis of the subspace W in DL^n. In the standard incline linear combination of the basis vector c_i, the ith coefficient of c_i is 1.

Proof

Since $c_i \; \varepsilon \; DL^n$, let $c_i = (c_{i1}, \; c_{i2},\ldots.,c_{in})$ for each $i =1$ to n. $c_i = \hat{x}_1c_1 + \hat{x}_2c_2+\ldots\ldots+\hat{x}_n c_n$ be the standard incline linear combination. This can be expressed as $xA = b$ where $b= (c_{i1},c_{i2},\ldots,c_{in}) \; \varepsilon \; DL^n$, $x = (\hat{x}_1, \hat{x}_2, \ldots.., \hat{x}_n)$

and

$$A = \begin{pmatrix} c_{11} \; c_{12} \; \ldots.. \; c_{1n} \\ c_{21} \; c_{22} \; \ldots.. \; c_{2n} \\ \therefore \qquad \ldots. \\ \vdots \\ c_{i1} \; c_{i2} \; \ldots.. \; c_{in} \\ \therefore \qquad \ldots.. \\ \vdots \\ c_{n1} \; c_{n2} \; \ldots. \; c_{nn} \end{pmatrix} \varepsilon \, DL_n$$

Let us find the maximum solution of the Theorem (5.1.6) incline relational equation $xA=b$ by using (5.1.2). The maximum solution $\hat{x} = (\hat{x}_1, \hat{x}_2,\ldots\ldots\ldots\ldots\ldots.. \hat{x}_n)$ is determined

by $\quad \hat{x} = min_{k \, \varepsilon K} \, \sigma(a_{jk}, b_k)$

$$\hat{x}_i = min\,\{\sigma(a_{i1},b_1), \sigma(a_{i2}, b_2), \sigma(a_{i3},b_3)\ldots..\sigma(a_{in},b_n)\}$$

$$= min\,\{\sigma(c_{i1},c_{i1}), \sigma(c_{i2},c_{i2})\ldots\ldots\ldots\ldots.. \sigma(c_{in},c_{in})\}$$

$$= min\,\{1,1,1,\ldots\ldots.1\} = 1$$

Therefore, $\hat{x} = (1, 1, \ldots.1)$,here 1 is the maximum element of £.

5.3. Applications in Automata Theory and Cryptography

In this section, we have discussed and highlighted an application of incline matrices in Automata Theory and Cryptography.

The Roll of Regular Matrices Over An Incline in Automata Theory

We have discussed the equivalence of finite state automata (DFSA) and the conversion of a NDFSA to an equivalent incline which is a DFSA, by way of constructing an incline structure with the set of states in the given NDFSA.

Definition 5.3.1 (Semi - Automata)

A finite state machine is said to be Semi - Automata, denoted as $M = \{S, I, O, f, s_0\}$. Where

S	-	Finite set of states
I	-	Finite set I of input symbols
O	-	Finite set O of output symbols
f	-	a transition function f that assigns a new state to every pair of state and input.
s_0	-	An initial state.

Definition 5.3.2 (Full Automata)

A finite state machine is said to be Full Automata, denoted as $M = \{S, I, O, f, g, s_0\}$. Where

S	-	Finite set of states
I	-	Finite set I of input symbols
0	-	Finite set 0 of output symbols
f	-	a transition function f that assigns a new state to every pair of state and input.
g	-	an output function g that assigns an output to every pair of state and input.
s_0	-	An initial state.

Lemma 5.3.3

Given any set S we can form an incline $£ = (\mathcal{P}(S), \cup, \cap)$, whose elements are subsets of S and the set "$\cup$" and "$\cap$" as an incline operation, which is a DFSA.

Proof

Consider any non deterministic full state automata $M = \{S, Q, \upsilon, \delta, F\}$

Where S - Finite set of states

Q - Finite set of inputs

υ - A function υ from $S \times Q \rightarrow S$, referred to as the transition function.

δ - A function δ from $S \times Q \rightarrow F$, referred to as the output function.

F - Final set of state.

The equivalent automata can be written as $M' = \{\mathcal{P}(S), Q', \upsilon', \delta', F'\}$

Where $\mathcal{P}(S)$ - the power set of S.

Q' - Q

υ' - $\mathcal{P}(S) \times Q \rightarrow S$

δ' - A function δ' from $\mathcal{P}(S)$ x $Q \rightarrow F'$

F' - F

Here, M' is a DFSA and M' satisfies the Definition of incline (1.2.11), hence M' is an incline.

Illustration 5.3.4

Consider the set $D = \{a,b,c\}$, $£ = (\mathcal{P}(S), \cup, \cap)$ is an incline, by Example (2.3.16), which is a DFSA.

Definition 5.3.5

A pair of matrices $A = (a_{ij})$ and $B = (b_{ij}) \varepsilon \pounds_{mn}$ are said to be space equivalent if and only if $\mathcal{R}(A) = \mathcal{R}(B)$ and $\mathcal{C}(A) = \mathcal{C}(B)$.

Example 5.3.6

First we construct the equivalent incline for the following NDFSA.

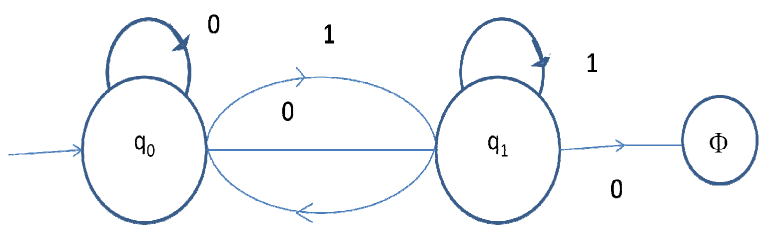
$M = < I = \{0,1\}, Q = \{q_0, q_1,\} \upsilon, \delta, s_0 = q_0, F = q_1\} >$

Here υ and δ is given by

υ	*0*	*1*
q_0	$\{q_0, q_1\}$	$\{q_1\}$
q_1	ϕ	$\{q_0, q_1\}$

δ	*0*	*1*
q_0	$\{q_1\}$	$\{q_1\}$
q_1	ϕ	$\{q_1\}$

The state diagram of the given NDFSA



The equivalent DFSA is represented by the equivalent incline

$M' = < I', \mathcal{P}(Q), \upsilon', \delta', q_0, F'>$

Where

$I' = I = \{0, 1\}.$

$\mathcal{P}(Q)$ - all possible subsets of states, that is, $\{\phi, [q_0], [q_1], [q_0, q_1]\}$

$q_0' = [q_0]$

$F' = F = \{q_1\}$

And since $\upsilon(q_0,0) = \{q_0,q_1\}$, we have $\upsilon'([q_0],0) = [q_0,q_1]$

Likewise,

$\upsilon'([q_0],1) = [q_1]$

$\upsilon'([q_1],0) = \phi$

$$\upsilon'([q_1],1) = [q_0,q_1]$$

$$\upsilon'(\phi,0) = \upsilon'(\phi,1) = \phi$$

Then $\quad \upsilon'([q_0,q_1],0) = [q_0,q_1]$

$$\upsilon(\{q_0,q_1\},0) = \upsilon(q_0,0) \cup \upsilon(q_1,0) = \{q_0,q_1\} \cup \phi = \{q_0,q_1\}$$

And $\quad \upsilon'([q_0,q_1],1) = [q_0,q_1]$

Since $\quad \upsilon'(\{q_0,q_1\},1) = \upsilon(q_0,1) \cup \upsilon(q_1,1)$

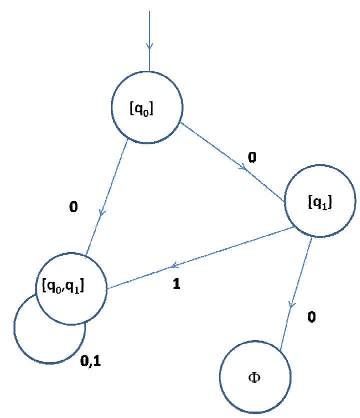
$$= \{q_1\} \cup \{q_0,q_1\} = \{q_0,q_1\}$$

Thus υ' and δ' is defined by

υ'	0	1
$[q_0]$	$[q_0,q_1]$	$[q_1]$
$[q_1]$	ϕ	$[q_0,q_1]$
$[q_0,q_1]$	$[q_0,q_1]$	$[q_0,q_1]$

δ'	0	1
$[q_0]$	$[q_1]$	$[q_1]$
$[q_1]$	ϕ	$[q_1]$
$[q_0,q_1]$	$[q_1]$	$[q_1]$

The state diagram for an incline given below:



Here, for the transition matrix $\upsilon, \upsilon', \delta$ and δ', $\mathcal{R}(\upsilon) = \mathcal{R}(\upsilon')$ and $\mathcal{R}(\delta) = \mathcal{R}(\delta')$, υ and δ are regular being idempotent, by Theorem (3.2.3), υ' and δ' are regular

Remark 5.3.7

Any NDFSA is equivalent to an DFSA and the corresponding transition matrices, that is, δ and δ' are space equivalent, υ and υ' are space equivalent.

Finite state Acceptor 5.3.8

An acceptor is a machine which can identify the strings of a language.

Example 5.3.9

Consider the set of non commuting words in *8* generators ordered by length and lexicographically when of equal length is a non commutative incline. Here the ordering is linear and $x+y$ is the greater of the two words that is, $x \geq y$ if the length of the word $x \geq$ the length of the word y

For this incline, let us construct a finite state acceptor that will accept the set of words multiples of *3*.

Let $M = <I,Q,q_0,\delta>$

Where $I = \{a,b,c,d,e,f,g,h\}$

$Q = \{q_0,q_1,q_2,q_3\}$

$F = \{q_0\}$

and δ is defined by

δ	x	y
q_0	q_1	q_2
q_1	q_3	q_3
q_2	q_3	q_3
q_3	q_0	q_0

Where $x = \{a,c,e,g\}$, $y = \{b,d,f,h\}$.

Consider the word *ad, age, decade, cabbage.*

Consider *ad*

We have $\delta(q_0,ad) = \delta(\delta(q_0,a)d)$

$= \delta(q_1,d) = q_3 \notin F.$

Hence *ad* is not accepted by *M*.

Consider *age*

$\delta(q_0, age) = \delta(\delta(q_0,a)ge)$

$= \delta(\delta(q_1,g)e)$

$= \delta(q_3,e) = q_0 \; \varepsilon F.$

Hence *age* is accepted by *M*.

Consider *decade*

$\delta(q_0, decade) \quad = \delta(\delta(q_0,d)ecade)$

$= \delta(\delta(q_2,e)cade)$

$= \delta(\delta(q_3,c)ade)$

$= \delta(\delta(q_0,a)de)$

$= \delta(\delta(q_1,d)\ e)$

$= \delta(q_3,e)$

$= q_0 \varepsilon F$

Hence *decade* is accepted by *M.*

Consider *cabbage*

$\delta(q_0, cabbage) \quad = \delta(\delta(q_0,c)abbage)$

$= \delta(\delta(q_1,a)bbage)$

$= \delta(\delta(q_3,b)bage)$

$= \delta(\delta(q_0,b)age)$

$= \delta(\delta(q_3,a)ge)$

$= \delta(\delta(q_0,g)e)$

$= \delta(q_1,e)$

$= q_3 \notin F.$

Hence *cabbage* is not accepted by *M.*

An Incline commitment Scheme with McEliece's Cipher

In this section, we have discussed the application of incline matrix in Cryptography for encryption and decryption. Many researchers have discussed fuzzy commitment scheme based on McEliece Scheme [3,4,25,26]. Just for sake of completeness we shall make use of the following definitions and procedure for encryption and decryption. For that, we consider a McEliece public key over an incline $£ = \{[0,1], +, \cdot\}$, where the addition is maximum and usual multiplication and 1 is the multiplicative identity as well as the greatest element and $£$ is not a Fuzzy algebra (Remark 5.1.18).

Crisp Commitment Schemes 5.3.10

In a commitment scheme, Rishi (sender) aim to entrust a concealed message m to Julie (receiver), intuitively a commitment scheme may be seen as the digital equivalent of a sealed envelope.

If sender wants to commit to some message m he just puts it into the sealed envelope, so that whenever sender wants to reveal the message to receiver, he opens the envelope. First of all the digital envelope should hide the message from Julie should be able to learn m from the commitment. Second the digital envelope should be binding; meaning with this that Rishi cannot change his mind about m, and by checking the opening of the commitment one can verify that the obtained value is actually the Rishi had in mind originally.

The McEliece Public –key Cryptosystem 5.3.11

Secret key : S is a random $(m \times m)$ non singular matrix over $£$, called the scrambling matrix. $£$ is an incline under the order relation "$\leq$", defined as an operation addition as supremum and multiplication as usual respectively. Where additive identity 0 is the least element and multiplicative identity "$1_£$" is the maximal element. T is a $(m \times n)$ generator matrix of binary Goppa code T with the capability of correcting an n-bit random error vector of weight less than (or) equal to a, and P is a random $(n \times n)$ permutation matrix.

Public Key $V = STP$

Encryption : $c = mV+e$, where m is a *n-bit* message, c is *n-bit* cipher text, and e is an n-bit random error vector of weight a.

Decryption: The receiver first calculates

$c' = cP^{-1} = mV\,ST+ep^{-1}$

Where P^{-1} is the inverse of P. Because the weight of eP^{-1} is the same as the weight of e, the receiver uses the decoding algorithm of the original code T to obtain $m' = mS$. Finally, the receiver recovers m by computing $m = m'S^{-1}$, where S^{-1} is the inverse of S.

Definition 5.3.12

An incline £ is metric space if there exists a function $d(x,y)$ on a set S defined as $d : £ \times £ \rightarrow R_0^+$ such that

(i) $d(x,y) = 0$ iff $x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,y)+ d(y,z) \geq d(x,z)$

(iv) *If* $x \leq y \leq z$ then $sup\,\{d(x,y), d(y,z)\} \geq d(x,z)$

(v) The operations $(+, \cdot)$ are continuous in the metric d.

Definition 5.3.13

Let $C\{0,1\}^n$ be a code set which consists of a set of code words c_i of length n. The distance metric between any two code words c_i and c_j in C is defined by

$dist\,(c_i,c_j) = \sum_{r=1}^{n} |c_{ir} - c_{jr}|,\ c_i,c_j\ \varepsilon\, C$

Definition 5.3.14

An error correction function f for a code C is defined as

$f(c_i) = \{c_j / dist\,(c_i, c_j)$ is the minimum, over $C/\{c_i\}\}$

Here, $c_j = f(c_i)$ is called the nearest neighbor of c_i.

Definition 5.3.15

The measurement of nearness between two code words c and c′ is defined by nearness $(c,c') =$ dist $(c,c')/n$.

It is obvious that $0 \leq$ nearness $(c, c') \leq 1$.

Definition 5.3.16

A function for a codeword c' to be equal to a given c is defined as

$$f(c') = \begin{cases} 0 & \text{if nearness } (c,c') = z \leq z_0 < 1, \\ z & \text{otherwise} \end{cases}$$

Incline commitment scheme with McEliece scheme 5.3.17

First select secret key S is a random $(m \times m)$ non singular matrix over£, called the scrambling matrix. T is a $(n \times n)$ generator matrix of a binary Goppa code T with the capability of correcting n-bit random error vector of weight less than or equal to a, and P is a random $(n \times n)$ permutation matrix.

Public Key : $V = STP$

A tuple $\{Q,H,M,F\}$ where $M \subseteq \{0,1\}^k$ is a message set which consider as a code, Q is a set of individuals, generally with three elements A as the committing party, B as the party to which commitment is made and TC as the trusted party, f is error correction function and $H = \{t_i,a_i\}$ are called the events occuring at times t_i, $i = 0, 1,2,$ as per algorithm a_i, $i = 0,1,2$. The secheme always culminates in either acceptance or rejection by A and B.

In the setup phase, the environment is setup initially and public commitment key CK generated, according to the algorithm setupalg (a_0) and published to the parties A and B at time t_0. During the commit phase, Rishi commits to a message $m \ \varepsilon M$ then he finds $g : m \rightarrow mV$.

Encryption : $E = mV+e$, where m is the k-bit message, E is an n-bit cipher text and e is an n-bit random error vector of weight a and + denotes modulo 2.

According to the algorithms commitalge1 into string c i.e. his commitment.

$c = commitalg\ (XOR,\ g(m),E),$

then after he sends c to Julie , which Julie will receive as $t(c)$, where t is the transmission function which includes noise.

In the open phase,Rishi sends the procedure for revealing the hidden commitment at time t_2 and use this.

So Rishi discloses the procedure *g(m)* and *E* to Julie open the commitment.

openalg (e_2) :

Julie construts *c'* using commitlag, message *t(m)* and opening key i.e.

$c' = commitalg\ (XOR,\ t(g(m)),\ t((E))$

and checks whether the result is same as the received commitment *t(c).*

Final Decision making.

If (nearness $(t(k), f(c')) \leq Z_0)$

Then *A* is bound to act as in *m*

Else he is free not to act as *m*.

Then after acceptance, Julie calculates $f(c')\ (STP)^{-1}$ and finally gets the message.

Working Process for Illustration 5.3.18

Secret key: *S* is a random (4 × 4) nonsingular matrix over £, called the scrambling matrix, *T* is a (4 × 7) generator matrix of binary Goppa code T with the capability of correcting an 7 - bit random error vector of weight less than or equal to *a*, and *P* is a random (7 ×7) permutation matrix.

Public Key : $V = STP$

Encryption : Let g $: m \rightarrow mV,$ where *m* is a 4 -bit message. Then after for the sake of secrecy add error *e*, which is a 7-bit random error vector of weight *a,* then $E = g(m) + e,$ *E* is a 7-bit ciphertext.

Now commitment,

$c = commitalg\ \ (CK,\ g(m),E)$

Decryption: The receiver first calculates

$c' = commitalg\ (CK,t(g(m)),\ t\ (E)),$ where *t* is the transmission function. The receiver checks the dist $(t(c),c') \neq 0,$ then apply error correction function *f* to *c'* and finds *f(c').* Then after apply decision making:

If (nearness $(t(c), f(c') \leq Z_0))$

Then A is bound to act as in m.

Else she is free not t act as m.

The receiver uses the decoding algorithm of the original code T to obtain $m' = m\ STP$

Finally, the receiver recovers m by computing $m = m'(STP)^{-1}$, where $(STP)^{-1}$ is the inverse of STP.

Illustration 5.3.19

Let D = {Rishi, Julie}

That is, we consider a situation where there is not trusted party.

Message space:

Let $M = \{0000, 1011, 0101, 1110, 1010, 1100, 1111\} \subset \{0,1\}^4$.

Message:

Let m = 1001.

Encoding function:

Let

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$g(m) = mSTP = 1011000$

$E = g(m) + e = 1111100$

$c = commitalg(\ g(m)\ XOR\ E) = 0100100$

Let the transmitted value $t(c) = 1000100$, which includes noise.

Rishi discloses the procedure $g(m)$ and E to Julie to open the commitment.

Suppose Julie gets $t(g(m)) = 1011010$ and $t(E) = 0110110$

Julie compute

$$c' = commitalg(\ t(g(m)) XOR\ t(E)) = 1101100$$

Julie apply the error correction function f to c': $f(c') = 1101100$

Julie accepted $t(c) = f(c') = 1101100$

Finally Julie calculate $f(c')\ (STP)^{-1} = 1101100 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = 1001.$

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CONCLUSION AND SCOPE

In the present investigation on matrices over an incline we have studied and discussed the relation between various combinations of the following classifications of an incline:

(i) Regular / Non Regular (ii) Commutative / Non Commutative

(iii) Whose elements are all linearly ordered / Whose elements are not linearly ordered

(iv) With multiplicative identity / Without multiplicative identity

We have exhibited that, in an incline £, under the order relation "$\leq$" defined as $x \leq y \Leftrightarrow x + y = y$, the greatest element '*1*' is the multiplicative identity of the elements of *DL*, the set of idempotent elements in the incline £. In particular, for a regular incline £, *DL* coincides with £ and therefore '*1*'coincides with the multiplicative identity "$1_£$" of £. On the other hand, for any incline £, that has the multiplicative identity "$1_£$", it coincides with the greatest element '*1*' and the converse need not be true. We have illustrated this with an example.

We have exhibited that a regular incline £ whose elements are all linearly ordered, is a commutative incline and the operations on £ reduce to the max-min compositions. Thereby, in the present work we have considered the incline with zero element – '*0*' and greatest element -'*1*' and which has no multiplicative identity.

We have proved that every finite subspace generated by the linearly ordered idempotent elements in an incline has a unique standard basis. In particular, for a Fuzzy algebra with support *[0,1]* under the operation max-min (or min - max), the result reduce to the result of Kim and Roush. It is known that, for a regular element $a\{1,2\}= a$. This need not be the case for matrices over an incline. On this basis, we have discussed the existence and construction of various g-inverse for a matrix over an incline whose idempotent elements are linearly ordered. We have established equivalent conditions for a matrix over an incline whose idempotent elements are linearly ordered to be regular. Cao, Kim and Roush have obtained an algorithm to determine the regularity and g-inverse of matrices over an incline, in which idempotent elements are linearly ordered. In the present work, we have provided an algorithm for matrices over a regular incline *£* whose elements are not linearly ordered.

We have determined conditions for the incline relational equations of the form $xA = b$, where A is an incline matrix and b is a vector over an incline to be consistent. Our results on incline relational equations can be applied to linear complimentarily problems in Control Theory, by extending the equation $xA = b$, where A is a block matrix and using the Schur complements of it's blocks.

We have exhibited that, any set S can be represented as an incline with set union and intersection as incline operations. By way of constructing an incline structure with the set of states in the given NDFSA. We have discussed the equivalence of two finite state machines in terms of equality of the spaces associated with the corresponding transition matrices. We have highlighted the role of incline matrices in the determination of encryption and decryption in Cryptography. These techniques can be extended to inclines of other algebraic structures.

Our results on regular incline matrices lead to the concept of transitive closures and in this direction the spectral properties, index and periods of inline matrices can be developed. There is a scope for further study on matrices over on regular non commutative incline whose elements are linearly ordered and for commutative non regular incline whose elements are not linearly ordered.

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1. **AR.Meenakshi, P.Shakila Banu,** *g-Inverses Of Matrices Over A Regular Incline,* **Advances in Algebra, 3 (2010) 33-42.**
2. **AR.Meenakshi, P.Shakila Banu***, Matrices Over A Regular Incline*, **The International Journal of Computational Cognition,** USA (to appear)
3. **AR.Meenakshi, P.Shakila Banu**, *On Regularity Of Incline Matrices*, (Communicated).
4. **AR.Meenakshi, P.Shakila Banu,** *Incline Relational Equations*, (Communicated).
5. **AR.Meenakshi, P.Shakila Banu,** *Standard Linear Combination Of Incline Vectors* (Communicated).

Papers Presented in the Conferences:

1. **P.Shakila Banu**, *"Matrices Over an Incline",* 1^{st} Annual Research Congress (KUARC), Karpagam University, Dec. 7-10, 2009.
2. **P.Shakila Banu**, *"Indices and Periods of Incline Matrices",* National Conference on Discrete Mathematics Algebra and their Application, Department of Mathematics, Karpagam University, Dec 22^{nd} and 23^{rd}, 2009.
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4. **P.Shakila Banu**, *"Regular Incline Matrices with Applications in Cryptography"* Recent Trends in Mathematics and its Applications, Alagappa University, Karaikudi, Dec 22^{nd}, 2010.