1. INTRODUCTION

1.1 Review of literature

Initially fuzzy set theory was proposed by Zadeh [46] as a means of representing mathematically any imprecise (or) vague system of information in the real world and for the purpose of developing expert systems and soft computing. The mathematical formulation and the membership function of a fuzzy set was introduced by Zadeh in 1965[46] as a generalization of a crisp set. Thus the membership function μ_A by which a fuzzy set A is defined has the form $\mu_A : X \to [0, 1]$. For recent development and application of fuzzy theory one can refer [9, 22, 47].

Matrices play an important role in various areas in science and engineering to represent any binary relation. However, we cannot successfully use traditional classical matrices because of various types of uncertainties present in real world situations such as problems in economics, engineering, environment, social science, medical science etc. do not always involve crisp data. These types of problem are solved by using fuzzy matrix. Thus fuzzy matrices have been proposed to represent fuzzy relations. Henceforth by a fuzzy matrix, we mean a matrix over the fuzzy algebra $F = [0, 1]$ under the fuzzy operations $(+, .)$ defined as $a + b = max \{a, b\}$ and $a \cdot b = min \{a, b\}$ for all $a, b \in F$. Thomason [42] has introduced fuzzy matrices and discussed about the Convergence of powers of a fuzzy matrix. The theories of fuzzy matrices were developed by Kim and Roush [19] as an extension of Boolean matrices [18]. For more details on fuzzy matrices one may refer [35]

A matrix $A\in F_{mn}$, the set of $m\times n$ matrices over the fuzzy algebra F is said to be regular if there exists $X\in F_{nm}$ such that $AXA = A$ under the fuzzy operations. In this case, X is called a generalized inverse of A and is denoted by A^{-}. $A\{1\}$ denotes the set of all g- inverse of A. Each element $a\in F$ is regular, since $axa = a$ holds under the fuzzy multiplication for all $x \geq a$ ('$\geq$' is the usual ordering on real number). Hence F is regular. It is well known that [30], for arbitrary ring R, R is regular if and only if R_{mn}, the set of all $m\times n$ matrices over R is regular. However, this fails for fuzzy matrices under the operation induced by the fuzzy algebra. Regular matrices play an important role in many branches of mathematics, since the regularity condition is a linear condition that solves linear equations and takes the place of canonical decomposition.

A study on regular fuzzy matrices was initiated by Kim and Roush [19]. Since then, many researchers have studied on regular fuzzy matrices [6, 19, 33, 34, 36]. Kim and Roush [19] have shown that for a regular fuzzy matrix, the row rank and the column rank are equal. Further, every invertible matrix is regular. Thus regular fuzzy matrices are a generalization of invertible matrices. Kim and Roush have successfully established conditions for the existence of various generalized inverses of fuzzy matrices, similar to the study on matrices over a field and in particular for complex matrices [1]. Regular fuzzy matrices play an important role in estimation, inverse problem and fuzzy optimation problems. For more details on regular fuzzy matrices one may refer [35].

Recently the concept of Interval valued Fuzzy matrices (*IVFM*) as a generalization of fuzzy matrix was introduced and developed by Shyamal and Pal [41], by extending the max.min operation in fuzzy algebra $F = [0, 1]$ for elements $a, b \in F$, $a + b = max\{a, b\}$ and $a \cdot b = min\{a, b\}$. The concept of interval valued fuzzy matrix is one of the recent topics developed for dealing with the uncertainties present in most of our real life situations. The parameterization tool of interval valued fuzzy matrix enhances the flexibility of its applications. This motivated us to study on regular interval valued Fuzzy Matrices (*IVFM*). In this present investigation, we have represented an *IVFM* $A = (a_{ij}) = ([\, a_{ijL} , a_{ijU}\,])$ where each a_{ij} is a subinterval of the interval $[\, 0, 1]$, as the interval matrix denoted as $A = [A_L , A_U]$ whose ij^{th} entry is the interval $[\, a_{ijL} , a_{ijU}\,]$, a_{ijL} and a_{ijU} are the ij^{th} entries of the lower limit $A_L = (\, a_{ijL}\,)$ and upper limit $A_U = (\, a_{ijU}\,)$ respectively. Both A_L and A_U are fuzzy matrices. Clearly $A_L \leq A_U$, we have determined the structure of row space and column spaces of regular *IVFM*, We observe that the Invertible *IVFM* reduces to invertible fuzzy matrices and we have discussed the existence and construction of *g*- inverses , $\{ 1, 2 \}$, $\{1, 3\}$ and $\{ 1, 4\}$ inverses of an *IVFM.*

The concept of Schur complement of a complex matrix goes back to Sylvester in 1981. The term Schur complement was coined by Haynsworth [14] in her studies on the inertia of partitioned Hermitian Matrices. Properties of Schur complements are used in computing inertias of matrices, covariance matrices of conditional distributions and in the study of restrictions of linear operator to subspace. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A non – Singular the Schur complement of A in M is the matrix $D - CA^{-1} B$, where A^{-1} is inverse of A. For a matrix M over any field, partitioned in the form $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A rectangular (or) singular, a generalized Schur complement of A in M is defined as $S = D - CA^{-}B$, where A^{-} is an $\{1\}$ – inverse of A. Marsaglia and Styan [29] investigated that under certain conditions S is independent of the choice of A^{-}. In 1970, Meyer [31, 32] explored various generalized inverses of triangular, block triangular matrices over the field of complex numbers. In 1974, Burns et al. [3] established formula for various generalized inverses of a partitioned complex matrix under certain conditions involving Schur complements.

Recently the concept of Schur complement is extended to fuzzy matrices by Meenakshi [33, 34]. Necessary and sufficient conditions are determined for regularity of a block fuzzy matrix and for the Schur complement of a block fuzzy matrix to be regular. Further, she has discussed characterization of idempotency of a class of triangular Toeplitz matrices. Toeplitz fuzzy matrices arise in scientific computing and engineering field. In our study, we derive a set of equivalent conditions for the regularity of Interval valued block fuzzy matrices (*IVBFM*). We have obtained generalized inverse formulae for an *IVBFM* and derive equivalent conditions for the idempotency of a triangular Toeplitz *IVBFM* as a generalization of results found in [34].

The notion of fuzzy relational equations based upon the max.min composition was first proposed and investigated by Sanchez [38]. Since then many researchers have developed the study on fuzzy relational equations [13, 16, 38, 47]. In [6], Cho has established the fuzzy relational equation of the form $xA = b$ is consistent when A is a regular fuzzy matrix and bX is a solution for some g- inverse X of A. Hence the set of all solutions $\Omega\,(A,\ b)$ of $xA= b$ is non empty when A is regular. A Method of determining minimum solution of fuzzy relational equations are provided in [16]. It is shown that $\Omega\,(A,\ b)$ has unique maximal element [47] and (p. 98 [22]), however the minimal element is not unique. Guo et al. [13] studied the minimal and maximal element of $\Omega\,(A,\ b)$. A new algorithm is proposed to solve the fuzzy relational equations in [25]. Fuzzy relational equations play an important role in fuzzy set theory and fuzzy logic systems, from both of the theoretical and practical viewpoints. The notion of fuzzy relational equations is associated with the concept of composition of binary relations.

In [43] consistency of system of interval valued fuzzy relational equation $xA^I = b^I$ where A^I is the set of fuzzy matrices $\{ A' / A^- \leq A' \leq A^+\}$ and $b^I = \{ b'/ b^- \leq b' \leq b^+ \}$ is discussed and complete set of solutions $for\ b^- \leq xA' \leq b^+$ is determined. This solution set was restructured by Guangzhi Li and Fang [12]. Later interval valued fuzzy relational equation were developed by [27, 45]. In the present work, we discuss the consistency of Interval valued fuzzy relational equations, equivalent condition for the existence of interval maximum solution of $xA = b$ for $A \in (IVFM)$ is determined and a new algorithm is proposed to solve the interval valued fuzzy relational equations. We derive the equivalent conditions for consistency of Interval valued block fuzzy relational equation of the form $xM = b$ where M is a Interval valued block fuzzy matrix as an extension of the result found in [33].

Kim and Roush [19] has established that every fuzzy matrix has an index and period that is, for $A \in F_n$ a fuzzy matrix of order n, then the least $k > 0$ such that $A^{k+d} = A^k$ is called the index of A, the least $d > 0$ such that $A^{k+d} = A^k$ is called the period of A. Further, for $A \in F_n$, there exists a positive integer $p \leq n$-1, such that $A^p = A^{p+1} = A^{p+2}$ and A^p is called the transitive closure of A. The transitive closure of the relation matrices, relevance matrices and document descriptor matrices reveal more accurate results for the system's user in document retrieval systems based on concept network and extended fuzzy concept network. In recent years, several fuzzy information retrieval method based on fuzzy set theory [46] have been proposed for improving the disadvantage of the Boolean logic model such as [15, 23, 17, 37].

In [28], Lucarella et al. presented a fuzzy information retrieval system (FIRST) based on concept networks. Many researchers have presented techniques to deal with document retrieval techniques using knowledge based fuzzy information techniques [4, 5, 9, 44] and ([35] pp.221 – 226). In the present study, we determine the index and period an *IVFM* in terms of that of its lower and upper limit fuzzy matrices, which leads to the definition of transitive closure of the concept *IVFM*. As an application, we highlight the role of the transitive closure of the *IVFM* in Document retrieval systems.

De et al. [10] have studied Sanchez's [38, 39] method of medical diagnosis using intuitionistic fuzzy set. Saikia et al. [40] have extended the method in [10] using intuitionistic fuzzy set theory. In [8], Chetia and Das have studied Sanchez's approach of medical diagnosis through interval valued fuzzy soft sets obtaining an improvement of the same presented in De et al. [10, 40]. Here we have extended Sanchez's approach for medical diagnosis using representation of an *IVFM* as an interval matrix of two fuzzy matrices.

In graph theory the shortest path problems is the problem of finding a path between two vertices (or nodes) such that sum of the weight of its constituent edges is minimized. An example is finding the quickest way to get from one location to another on a road map; in this case the vertices represent locations and weight on each edge represents the time needed to travel that segment. The shortest path problem has transportation, communication routing and scheduling. Now, in any network path the arc length may represent time or cost. Therefore in real world, it can be considered to be a fuzzy set. We consider a directed network consisting of a finite set of vertices and finite set of directed edges. It is assumed that there is only one directed edge between any two vertices. The fuzzy shortest path problems were first analyzed by Dubois and Prade [11]. They used Floyd's algorithm and Ford's algorithm to treat the fuzzy shortest path problem. Although in their method the shortest path length can be obtained, may be the corresponding path in the network doesn't exist. Klein [21], proposed a dynamical programming and later developed by many researchers [2, 7, 20, 24, 26]. In this thesis, we propose a new approach to determine the shortest path in interval valued fuzzy network (*IVFN*) in which the edges representing the roads connecting the cities and edges $(i, i+1)$ has an associated weight representing the traffic on the road connecting the cities i and $i+1$, which is an interval fuzzy number of the form $R_i = [R_{iL}, R_{iU}]$ for each i.

In the present investigation, we have represented an IVFM $A = (a_{ij}) = ([a_{ijL}, a_{ijU}])$ where each a_{ij} is a subinterval of the interval $[0, 1]$, as the interval matrix $A = [A_L, A_U]$ whose ij^{th} entry is the interval $[a_{ijL}, a_{ijU}]$, a_{ijL} and a_{ijU} are the ij^{th} entries of the lower limit $A_L = (a_{ijL})$ and upper limit $A_U = (a_{ijU})$ respectively. Clearly A_L and A_U are fuzzy matrices, such that $A_L \leq A_U$. We have discussed the regularity of an *IVFM* in terms of those of its lower limit and upper limit fuzzy matrices as generalization of regular fuzzy matrices available in the literature [6, 19, 35]. We have proved that the row space and column spaces of an *IVFM* are preserved under the interval matrix representation. This leads to the structure of set of all g-inverses of a regular *IVFM*. We extend the concept of Schur complement for an *IVFM* and obtained the set of equivalent condition for regularity of Interval Valued Block Fuzzy Matrices as a generalization of the results on regular block fuzzy matrices found in [33, 34, 35, 36]. We have discussed the consistency of the interval valued fuzzy relational equation as a generalization of fuzzy relation equation studied in [35, 38] and determine the complete set of solution of $xA = b$ where A is an *IVFM* and b is an IVF Vector. As an application, we have highlighted the role of *IVFM* in Document Retrieval System, Medical Diagnosis and Shortest path Network with suitable illustration.

1.2 Notations, Basic Definitions and Preliminaries

In this section, the notations, some basic definitions and relevant results needed elsewhere in this thesis are introduced.

Let *IVFM* denotes the set of all interval valued fuzzy matrices, that is, fuzzy matrices whose entries are all subintervals of the interval [0, 1].

For $A \in (IVFM)_{mn}$,

A^T – the transpose of A

A_{i*} – the i^{th} row of A

A_{*j} – the j^{th} column of A

$\mathcal{R}(A)$ – the row space of $A = \{y \in F_m / y = xA\}$

$\mathcal{C}(A)$ – the column space of $A = \{y \in F_n / y = Ax\}$

$\rho_r(A)$ – the row rank of A

$\rho_c(A)$ – the column rank of A

A^- – the g – inverse of A, that is $AA^-A = A$.

$A\{1\}$ – the set of all g- inverses of A

N – the set $\{1, 2, ..., n\}$

$\Omega(A, b)$ – the solution set of $xA = b$, where $A \in (IVFM)_{mn}$ and $b \in (IVFV)_n$

Definition 1.2.1

For $A = (a_{ij})$ and $B = (b_{ij}) \in F_{mn.}$

$A + B = (max\ (a_{ij}, b_{ij}))$.

$AB = (max_k\ (min\ (a_{ik}, b_{kj})))$.

Lemma 1.2.2

For $A, B \in F_{mn}$, we have the following:

(i) $\mathcal{R}(B) \subseteq \mathcal{R}(A) \Leftrightarrow B = XA$ for some $X \in \mathcal{F}_m$

(ii) $\mathcal{C}(B) \subseteq \mathcal{C}(A) \Leftrightarrow B = AY$ for some $Y \in \mathcal{F}_n$.

Lemma 1.2.3

For $A \in \mathcal{F}_{mn}, B \in \mathcal{F}_{np}$, we have the following:

(i) $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$

(ii) $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$

Lemma 1.2.4 (Lemma 2 [34])

For $A, B \in F_{mn}$, if A is regular, then

(i) $\mathcal{R}(B) \subseteq \mathcal{R}(A) \Leftrightarrow B = BA^{-}A$ for each $A^{-} \in A\{1\}$

(ii) $\mathcal{C}(B) \subseteq \mathcal{C}(A) \Leftrightarrow B = AA^{-}B$ for each $A^{-} \in A\{1\}$.

Lemma 1.2.5

If $A \in F_{mn}$ with $\mathcal{R}(A) = \mathcal{R}(A^TA)$, then A^TA is regular fuzzy matrix if and only if A is a regular fuzzy matrix. If $A \in F_{mn}$ with $\mathcal{C}(A) = \mathcal{C}(AA^T)$, then AA^T is a regular fuzzy matrix if and only if A is a regular fuzzy matrix.

Definition 1.2.6

A matrix $A \in F_n$ is said to be invertible if and only if there exists $B \in F_n$ such that $AB = BA = I_n$.

Lemma 1.2.7

Let $A \in F_n$. A is invertible if and if only if A is a permutation matrix. In this case the transpose A^T is the unique inverse of A.

Definition 1.2.8

Let $A \in F_{mn}$. A is regular if and only if A has a g-inverse. If A is regular, then a g-inverse of A is denoted as A^{-} and $A\{1\}$ is the set of all g-inverses of A satisfying $AA^{-}A = A$ for all $A^{-} \in A\{1\}$.

Lemma 1.2.9

Let $A \in F_{mn}$ with $\rho(A) = \rho_r(A) = \rho_c(A) = r$, then there exist matrices $B \in F_{mr}$ $C \in F_{rn}$, such that $A = BC$ with $\rho(A) = \rho_c(B) = \rho_r(C) = r$. This is called rank factorization of A.

Lemma 1.2.10 (Theorem 3.1[33])

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, A and D are regular block fuzzy matrices with $\mathcal{R}(C) \subseteq \mathcal{R}(A)$, $\mathcal{C}(B) \subseteq \mathcal{C}(A)$. Then the following are equivalent.

(i) $\mathcal{R}(B) \subseteq \mathcal{R}(D)$, $\mathcal{C}(C) \subseteq \mathcal{C}(D)$, the Schur complement M/A and M/D are Fuzzy Matrices.

(ii) M is regular, $BD^{-}C$ is invariant and $m = \begin{pmatrix} A^{-} + A^{-}BD^{-}CA & A^{-}BD^{-} \\ D^{-}CA^{-} & D^{-} \end{pmatrix}$

is a g- inverse of M, for some g- inverse A^{-} of A and D^{-} of D.

Lemma 1.2.11 (Corollary 3 [34])

For the lower block triangular matrix $M = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$, the following statements are equivalent:

(i) M is idempotent.

(ii) The blocks A and D are idempotent with $CA = DC = C$.

(iii) The blocks A and D are idempotent matrices with $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{C}(C) \subseteq \mathcal{C}(D)$.

Definition 1.2.12

Any element $\hat{x} \in \Omega(A, b)$ is called a minimal solution of $xA = b$, if $x \geq \hat{x}$ for all $x \in \Omega(A, b)$.

Definition 1.2.13

Any element $\check{x} \in \Omega(A, b)$ is called a maximal solution of $xA = b$, if $x \leq \check{x}$ for all $x \in \Omega(A, b)$.

Lemma 1.2.14

Let $xA = b$ where $A = (a_{\mathrm{ij}}) \in F_{mn}$, $b = (b_{ij}) \in F_{1n}$. If $max_j\ (a_{ij}) < b_k$ for some $k \in N_n$, then $\Omega(A, b) = \phi$.

Lemma 1.2.15

For the equation $xA = b$ where $x = [x_j / j \in N_m]$, $b = [b_k / k \in N_n]$ and $A \in F_{mn}$, $\Omega(A, b) \neq \phi$, if and only if $\hat{x} = [\hat{x}_j / j \in N_m]$, defined as $\hat{x}_j = min\ \sigma(a_{ik}, b_k)$

Where $$\sigma(a_{ik}, b_k) = \begin{cases} b_k, & if\ a_{ik} > b_k \\ 1, & \text{otherwise} \end{cases}$$

It is well known that [19], every fuzzy matrix has an index and period, that is, for $A \in F_n$, the least $k > 0$ such that $A^{k+d} = A^k$ is called the index of A, the least $d > 0$ such that $A^{k+d} = A^k$ is called the period of A, these are denoted as $i(A)$ and $p(A)$ respectively.

Lemma 1.2.16 [19]

For $A \in F_{n,}$ if $A^{k+d} = A^k$ for some k, $d > 0$, then $k \geq i(A)$ and $p(A)/d$.

Definition 1.2.17

An Interval Valued Fuzzy Matrix *(IVFM)* of order *mn* is defined as $A = (a_{ij})_{mn}$, where $a_{ij} = [a_{ijL}, a_{ijU}]$ the ij^{th} element of A is an interval representing the membership value. All the elements of an *IVFM* are intervals and all the intervals are the subintervals of the interval $[0, 1]$. Let A and B be any two *IVFM*s. The following operations are defined for any two element $x \in A$ and $y \in B$, where $x = [x_L, x_U]$ and $y = [y_L, y_U]$ are intervals in $[0, 1]$ such that $x_L < x_U$ and $y_L < y_U$.

(i) $x + y = [max\{x_L, y_L\}, max\{x_U, y_U\}]$

(ii) $x \cdot y = [min\{x_L, y_L\}, min\{x_U, y_U\}]$

Here we shall follow the basic operations on *IVFM* as given in [41].

For $A = (a_{ij}) = ([a_{ijL}, a_{ijU}])$ *and* $B = (b_{ij}) = ([b_{ijL}, b_{ijU}])$ of order *mn* their sum denoted as A + B is defined as,

$$A+B = (a_{ij} + b_{ij}) = ([(a_{ijL} + b_{ijL}), (a_{ijU} + b_{ijU})]) \quad \rightarrow (1.2.1)$$

For $A = (a_{ij})_{mn}$ and $B = (b_{ij})_{np}$ their product denoted as AB is defined as,

$$AB = \left[c_{ij}\right] = \left(\sum_{k=1}^{n} a_{ik}\, b_{kj} \right) \qquad i=1, 2, \ldots, m \text{ and } j=1, 2, \ldots, p$$

$$= \left([\sum_{k=1}^{n} (a_{ikL} \cdot b_{kjL}), \sum_{k=1}^{n} (a_{ikU} \cdot b_{kjU})] \right) \quad i=1, 2, \ldots, m \text{ and } j=1, 2, \ldots, p \quad \rightarrow (1.2.2)$$

$A \leq B$ if and only if $A+B = B$ if and only if $a_{ijL} \leq b_{ijL}$ and $a_{ijU} \leq b_{ijU}$ →(1.2.3)

In particular if $a_{ijL} = a_{ijU}$ and $b_{ijL} = b_{ijU}$ then (1.2.2) reduces to the standard *max.min* composition of Fuzzy Matrices [19, 35].

In [12], the set of solution for $xA' = b'$where

$A' \in A^I = \{ A'/A^- \leq A' \leq A^+ \}$ and b' $\in$ $b^I = \{ b'/b^- \leq b' \leq b^+ \}$ →(1.2.4)

is determined further, when the solution set is not empty, it is shown that $xA' = b'$has one maximum solution and a finite number of minimum solutions.

Lemma 1.2.18 (Theorem 1 of [12])

x is a solution of $xA' = b'$where $A' \in A^I = \{ A'/A^- \leq A' \leq A^+ \}$ and $b' \in b^I = \{ b'/ b^- \leq b' \leq b^+ \}$ if and only if $x \in x^I$ for some $x^I \in \Omega$ (A^I, b^I) where $x^I = \{ x'/x^- \leq x' \leq x^+ \}$.

Lemma 1.2.19 (Corollary 2 [12])

For the equation $xA^I = b^I$, where $A^I = \{ A'/A^- \leq A' \leq A^+ \}$, $b^I = \{ b'/b^- \leq b' \leq b^+ \}$ and the fuzzy relational inequality system $\begin{cases} xA^- \geq b^- \\ xA^+ \leq b^+ \\ x \in X \end{cases}$ have the identical solution set.

1.3 Motivation

- We have represented an Interval Valued Fuzzy Matrix (*IVFM*) $A = (a_{ij}) =([a_{ijL} , a_{ijU}])$ where each a_{ij} is a subinterval of the interval $[0, 1]$, as the interval matrix denoted as $A = [A_L , A_U]$ whose ij^{th} entry is the interval $[a_{ijL} , a_{ijU}]$, a_{ijL} and a_{ijU} are the ij^{th} entries of the lower limit $A_L = (a_{ijL})$ and upper limit $A_U = (a_{ijU})$ respectively. Both A_L and A_U are fuzzy matrices, satisfying the order relation $A_L \leq A_U$. This representation of an IVFM motivated us to introduce and develop the theory of Regular Interval Valued Fuzzy Matrices (*IVFM*) as a generalization of regular fuzzy matrices [6, 19, 35].

In our study on Regular Interval Valued Fuzzy Matrices, we have the following inclusion relation among the sets of Interval valued fuzzy matrices of order n denoted as $(IVFM)_n$.

$(IVFM)_n \supseteq$ Regular $(IVFM)_n \supseteq$ Invertible $(IVFM)_n \supseteq$ Permutation $(IVFM)_n$.

By using our interval matrix representation of an $(IVFM)_n$, we observe that no two permutation matrices are comparable. Hence we have the following:

Invertible $(IVFM)_n$ = Invertible Matrices in F_n = Permutation Matrices in F_n.

This motivated us to study on Regular $(IVFM)_n$, the associated g-inverses and their extension of block $(IVFM)_n$.

- In [12], the set of solution of $xA^I = b^I$, where $A^I = \{ A'/A^- \leq A' \leq A^+ \}$ and $b^I = \{ b'/b^- \leq b' \leq b^+\}$ are determined. This motivated us to compare the solution set of the Interval valued fuzzy relational equations xA = b where $A = [A_L, A_U] \in (IVFM)_{mn}$ and $b = [b_L, b_U] \in (IVFM)$ with that of $xA^I = b^I$ for $A^I = \{ A'/A_L \leq A' \leq A_U \}$ and $b^I = \{ b'/ b_L \leq b' \leq b_U\}$.

- We determine the index and period an *IVFM* in terms of that of its lower and upper limit fuzzy matrices, which leads to the definition of transitive closure of the concept *IVFM*. This motivated us to develop the transitive closure of the Interval valued fuzzy matrices in Document retrieval systems as a generalization of transitive closure of the fuzzy matrices in Document retrieval systems [35].

- Sanchez's approach for medical diagnosis [39], motivated us to study on Interval valued fuzzy matrices in Medical diagnosis using our representation of an *IVFM* as an interval matrix of two fuzzy matrices.

- ➢ The shortest path in interval valued fuzzy network (*IVFN*) in which the edges representing the roads connecting the cities and edge (i , $i+1$) has an associated weight representing the traffic on the road connecting the cities i and $i+1$, which is an interval fuzzy number of the form $R_i = [R_{iL} , R_{iU}]$ for each i. This motivated as to study the shortest path in an Interval valued fuzzy network (*IVFN*) as a generalization of fuzzy network studied in [21] by defining the shortest path for an *IVFN* as that path for which the shortest path in lower limit fuzzy network coincides with the shortest path in upper limit fuzzy network.

1.4 Summary of the results

A short account of the results obtained in this thesis is given.

Interval Valued Fuzzy Matrices (*IVFM*)

We have represented an *IVFM* $A = (a_{ij}) = ([a_{ijL} , a_{ijU}])$ where each a_{ij} is a subinterval of the interval [0, 1], as the interval matrix denoted as $A = [A_L , A_U]$ whose ij^{th} entry is the interval $[a_{ijL} , a_{ijU}]$, a_{ijL} and a_{ijU} are the ij^{th} entries of the lower limit $A_L = (a_{ijL})$ and upper limit $A_U = (a_{ijU})$ respectively. Both A_L and A_U are fuzzy matrices, such that $A_L \leq A_U$. We introduce the concept of regular interval valued Fuzzy Matrices (*IVFM*) as a generalization of regular Fuzzy matrices [6, 19, 35]. The structure of row space and column space of regular Interval valued fuzzy matrices are obtained. The regularity of an *IVFM* is discussed by representing an *IVFM* as an interval of a pair of fuzzy matrices and illustrated with suitable examples for a regular *IVFM*. We exhibit that the class of all invertible $(IVFM)_n$ coincides with the invertible matrices in F_n, then, as an application we show that the regularity of *IVFM* is preserved under congruence relation. In [41], the identity *IVFM* is defined as the *IVFM* in which all the diagonal entries are [1, 1] and all other entries are $[0, 0]$. It is denoted by I_n under the interval matrix representation of the identity IVFM is $[I_n, I_n] = I_n$. The results on regular *IVFM* form a part of the author's paper appeared in "*Advances in Fuzzy Mathematics*". We have determined the *g* – inverses of interval valued Fuzzy Matrices (*IVFM*) as a generalization of *g*-inverses of regular fuzzy matrices studied in [6, 19, 35]. The existence and construction of *g*- inverses, {*1, 2*} inverses, {*1, 3*} inverses and {*1, 4*} inverses of Interval valued fuzzy matrix are discussed in terms of the row and column spaces of *IVFM.* The contents form a part of the material of the author's paper accepted in "*ITB Journal of Science*".

Interval Valued Block Fuzzy Matrices (*IVBFM*)

We have introduced the concept of regularity for Interval Valued Block Fuzzy Matrices (*IVBFM*) as a generalization of regularity of block fuzzy matrices studied in [33] and as an extension of regularity of *IVFM.* By introducing the concept of Schur complement for an *IVFM*, equivalent conditions for regularity of *IVFM* are obtained and examples are provided wherever necessary. The contents form a part of the material of the author's paper appeared in "*Advances in algebra*". We have discussed the generalized inverses formulae for an Interval valued block fuzzy matrices. We have derived equivalent conditions for the idempotency of a triangular Toeplitz interval valued fuzzy matrices (IVFM) of order up to 3, then discussed the idempotency of a general triangular Toeplitz IVFM of order *k*.

Interval Valued Fuzzy Relational Equations

We have discussed the consistency of the Interval valued fuzzy relational equations as a generalization of fuzzy relational equations discussed in [38] and determine the complete set of solutions of $xA = b$ where A is an *IVFM* and b is an IVF Vector, equivalent condition for the existence of Interval maximum solution of $xA = b$ for $A \in$IVFM and its relation with the maximum solution of the System

$xA^I = b^I$ is determined, where $A^I = \{ A'/ A_L \leq A' \leq A_U\}$ and $b^I = \{ b'/ b_L \leq b' \leq b_U\}$. These results are based on the author's paper accepted in the "*International Journal of computational Cognition*". We have discussed the consistency of the interval valued block fuzzy matrix equation $xM = b$,

$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in (IVFM)$, $x = [y, z]$ and $b = [c, d]$ be partitions of x and b respectively in conformity with that of M. A new algorithm is proposed to solve the interval valued fuzzy relational equation $P\cdot Q = R$ with *max.min* composition and max product composition. The algorithm operates systematically and graphically on a matrix pattern to get all the solution of P. An example is given to illustrate its effectiveness.

Applications of Interval Valued Fuzzy Matrices (*IVFM*)

We have determined the index and period of an Interval valued Fuzzy matrix (*IVFM)* in terms of that of its lower and upper limit fuzzy matrices, which leads to the definition of transitive closure of the concept *IVFM.* We have discussed knowledge – based interval valued fuzzy information retrieval method based on concept interval networks by using the transitive closure of the *IVFM* and illustrated with suitable examples. We have extended Sanchez's approach for medical diagnosis using the representation of an interval valued fuzzy matrix as an interval matrix of two fuzzy matrices. In our method the matrix operations involved are *max.min*, which is uniform and simpler than that are found in [8, 10, 40] where intuitionistic fuzzy matrix operations are involved in the computation of medical knowledge. We have introduced the arithmetic mean (*am*) matrix of an *IVFM* as the average of the lower and upper limit matrices A_L and A_U and directly apply Sanchez's method of medical diagnosis for the *am* (*A*)*,* which is a fuzzy matrix. The contents form a part of the material of the author's paper accepted in "*International Journal of Mathematical Analysis*". We proposed a new approach to determine the shortest path in Interval valued fuzzy network (*IVFN*) in which the edges representing the roads connecting the cities and each edge (i, $i+1$) has an associated weight representing the traffic on the road connecting the cities i and $i+1$, which is an interval fuzzy number of the form $R_i = [\, R_{iL}, R_{iU}]$ for each i and we apply the techniques used in [2, 21] to determine the shortest path in lower and upper limit of the fuzzy networks. We have defined the shortest path for an *IVFM* as that path for which the shortest path in lower limit fuzzy network coincides with the shortest path in upper limit fuzzy network and weight = (w_L , w_U) where w_L and w_U are the weights of the shortest path for lower and upper limit fuzzy networks respectively

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2. INTERVAL VALUED FUZZY MATRICES (*IVFM*)

In this chapter, first we represent an *IVFM* as an interval matrix of two fuzzy matrices. We define regular interval valued Fuzzy Matrices as a generalization of regular Fuzzy Matrices and determine the structure of Row space, Column space of an *IVFM* are obtained. We exhibit that the class of all invertible $(IVFM)_n$ coincides with the invertible matrices in F_n, then, as an application, we show that the regularity of *IVFM* is preserved under congruence relation. In [41], the identity *IVFM* is defined as the *IVFM* in which all the diagonal entries are $[1, 1]$ and all other entries are $[0, 0]$. It is denoted by I_n. By using representation of the identity *IVFM* is $[I_n, I_n] = I_n$ and also we shall discuss the g- inverses of an *IVFM* and their relations in terms of the Row and Column spaces of the matrix as a generalization of the results available in the literature on Fuzzy Matrices [19, 35] as a development of Regular *IVFM* and analogous to that for complex matrices [1].

2.1 Regular *IVFM*

In this section, we define regular interval valued Fuzzy Matrices as a generalization of regular Fuzzy Matrices and obtain the structure of Row space, Column space of an *IVFM* .

Definition 2.1.1

For a pair of Fuzzy Matrices $E = (e_{ij})$ and $F = (f_{ij})$ in F_{mn} such that $E \leq F$, let us define the interval matrix denoted as $[E, F]$, whose ij^{th} entry is the interval with lower limit e_{ij} and upper limit f_{ij}, that is $([e_{ij}, f_{ij}])$. Clearly $[E, F]$ is an *IVFM*. In particular for $E = F$, *IVFM* $[E, E]$ reduces to $E \in F_{mn}$.

For $A = (a_{ij}) = ([a_{ijL}, a_{ijU}]) \in (IVFM)_{mn}$, let us define $A_L = (a_{ijL})$ and

$A_U = (a_{ijU})$ clearly A_L and A_U belongs to F_{mn} such that $A_L \leq A_U$ and from Definition (2.1.1) A can be written as $A = [A_L, A_U]$ → (2.1.1)

Henceforth, we shall use this representation for an *IVFM.*

Lemma 2.1.2

For $A = [A_L, A_U] \in (IVFM)_{mn}$ and $B = [B_L, B_U] \in (IVFM)_{np}$, the following hold.

(i) $A^T = [A_L^T, A_U^T]$

(ii) $AB = [A_LB_L, A_UB_U]$

(iii) $A + B = [A_L+B_L, A_U+B_U]$

(iv) $[\alpha, \beta](A + B) = [\alpha A_L+\beta B_L, \alpha A_U+\beta B_U]$ for $\alpha \leq \beta \in F$

(iv) $[\alpha, \beta]\, A = [\alpha A_L, \beta A_U]$ for $\alpha \leq \beta \in F$

(v) $\alpha A = [\alpha A_L, \alpha A_U]$ for $\alpha \in F$

Proof

(i) This directly follows from the Definition (2.1.1)

(ii) Let $A = [A_L, A_U] = ([a_{ijL}, , a_{ijU}])$

$B = [B_L, B_U] = ([b_{ijL}, b_{ijU}])$

$$\text{Then } AB = \left([\sum_{k=1}^{n} (a_{ikL} \cdot b_{kjL}), \quad \sum_{k=1}^{n} (a_{ikU} \cdot b_{kjU})\,] \right) \qquad \text{(By (1.2.2))}$$

$= \quad [A_LB_L, A_UB_U]$

A_LB_L and A_UB_U are the *max.min* composition of matrices in F_{mp}.

(iii) Let $A = [A_L, A_U] = ([a_{ijL}, a_{ijU}])$

$B = [B_L, B_U] = ([b_{ijL}, b_{ijU}])$

$A + B = (a_{ij}+ b_{ij}) = [(a_{ijL}+ b_{ijL}), (a_{ijU}+ b_{ijU})]$ (By (1.2.1))

$= [A_L+B_L, A_U+B_U]$

(iii) Let $A = [A_L, A_U] = ([a_{ijL}, a_{ijU}])$

$B = [B_L, B_U] = ([b_{ijL}, b_{ijU}])$

$[\alpha, \beta](\alpha A + \beta B) = (\alpha a_{ij}+ \beta b_{ij}) = [(\alpha a_{ijL}+\beta\, b_{ijL}), (\alpha a_{ijU}+ \beta b_{ijU})]$

$= [\alpha A_L+\beta B_L, \alpha A_U+\beta B_U]$

(iv) Let $A = [A_L, A_U] = ([a_{ijL}, a_{ijU}])$

$[\alpha, \beta]\, A = [\alpha, \beta]\, [a_{ijL}, a_{ijU}]$

$= [\alpha(a_{ijL}), \beta(a_{ijU})]$

$= [\alpha A_L, \beta A_U]$ for $\alpha \le \beta \in F.$

(v) Let $A = [A_L, A_U] = ([a_{ijL}, a_{ijU}])$

$\alpha A = [\alpha(a_{ijL}), \alpha\, (a_{ijU})]$

$= [\alpha A_L, \alpha A_U]$ for $\alpha \in F$

Hence the Lemma.

Theorem 2.1.3

Let $A = [A_L, A_U] \in (IVFM)_{mn}$.

Then A is regular $IVFM \Leftrightarrow A_L$ and $A_U \in F_{mn}$ are regular.

Proof

Let $A \in (IVFM)_{mn}$

If A is regular $IVFM$, then by Definition (1.2.8) there exists $X \in (IVFM)_{mn}$ such that $A\,X\,A = A$

Let $X = [X_L, X_U]$ with $X_L, X_U \in F_{nm}$

Then by Lemma (2.1.2 (ii)),

$\Rightarrow$ $[A_L, A_U]\ [X_L, X_U]\ [A_L, A_U] = [A_L, A_U]$

$\Rightarrow$ $A_L X_L A_L = A_L$ and $A_U X_U A_U = A_U$

$\Rightarrow$ A_L is regular and A_U is regular F_{mn}.

Thus, A is regular $IVFM \Rightarrow A_L$ and $A_U \in F_{mn}$ are regular

Conversely,

Suppose A_L and A_U are regular, then $A_L X_L A_L = A_L$ and $A_U Y_U A_U = A_U$ for some X_L and $Y_U \in F_{mn}$. Hence, $X_L \in A_L\ \{1\}$. $Y_U \in A_U\ \{1\}$.

Since $A_L \leq A_U$, it is possible to choose atleast one $X \in A_L\ \{1\}$ and $Y \in A_U\ \{1\}$ such that, $X \leq Y$.

Let us define the interval valued Fuzzy Matrix $Z = [X, Y]$.

Then by Lemma (2.1.2 (ii)),

We get, $A\,Z\,A = [A_L, A_U]\ [X, Y]\ [A_L, A_U] = [A_L, A_U] = A$

Thus, A is regular.

Hence the Theorem.

In Theorem (2.1.3), both A_L and A_U to be regular is essential. This is illustrated in the following:

Example 2.1.4

Let us consider $A = [A_L, A_U] \in (IVFM)_3$ where $A_L = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$,

$A_U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Then $A_L \leq A_U$.

Here A_U is regular being idempotent and A_L is not regular by Algorithm 1 [19], there is no $X \in (IVFM)_3$ such that, $A_L XA_L = A_L$. Therefore A is not regular for, if A is regular, then for some $X \in (IVFM)_3$, $AXA= A \Rightarrow A_LX_LA_L \Rightarrow A_L$ is regular which is not possible.

Illustratation 2.1.5

Let us consider an interval matrix $[A, B]$ given by $A = \begin{pmatrix} 0 & 0 \\ 0.5 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0.5 & 0 \end{pmatrix}$.

Clearly both A and B are regular Fuzzy Matrices satisfying $A \leq B$.

By using Algorithm 1 of [19] we have,

$A\ \{1\} = \{X/X = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ b \geq 0.5,\ a, c, d, \in F\}$

$B\ \{1\} = \{Y/Y = \begin{pmatrix} 0 & b \\ 1 & d \end{pmatrix},\ b \geq 0.5,\ d, \in F\}$

$$X = \begin{pmatrix} 0 & 0.6 \\ 0.5 & 1 \end{pmatrix} \in A\,\{1\}, \quad Y = \begin{pmatrix} 0 & 0.7 \\ 1 & 1 \end{pmatrix} \in B\,\{1\}, \text{ clearly } X \leq Y.$$

Hence $[X, Y]$ is an *IVFM* satisfying

$[A, B]\,[X, Y]\,[A, B] = [A, B]$

Thus $[A, B]$ is regular.

Here $B\{1\} \subseteq A\{1\}$ but every element of $B\ \{1\}$ need not be comparable with any element of $A\{1\}$.

For, $$Y = \begin{pmatrix} 0 & .7 \\ 1 & 1 \end{pmatrix} \in B\ \{1\} \text{ is not comparable}$$

With $$X = \begin{pmatrix} .5 & .6 \\ .3 & .8 \end{pmatrix} \in A\ \{1\}$$

Further, $$X = \begin{pmatrix} .5 & .6 \\ .5 & .9 \end{pmatrix} \in A\ \{1\}$$

But $$X = \begin{pmatrix} .5 & .6 \\ .5 & .9 \end{pmatrix} \notin B\ \{1\}$$

Therefore, $A\ \{1\} \not\subset B\ \{1\}$.

Next we shall see that the spaces associated with an *IVFM* are preserved for its row and column spaces in the following:

Theorem 2.1.6

Let $A = [A_L, A_U]$ be an $(IVFM)_{mn}$

Then, (i) $\mathcal{R}(A) = [\mathcal{R}(A_L), \mathcal{R}(A_U)] \in (IVFM)_{1n}$

(ii) $\mathcal{C}(A) = [\mathcal{C}(A_L), \mathcal{C}(A_U)] \in (IVFM)_{1m}$

Proof

(i) Since $A \in (IVFM)_{mn}$, any vector $x \in \mathcal{R}(A)$ is of the form $x = yA$ for some $y \in (IVFM)_{1m}$ that is, x is an interval valued vector with n components.

Let us compute $x \in \mathcal{R}(A)$ as follows:

x is a linear combination of the rows of $A \Rightarrow x = \sum_{i=1}^{m} \alpha_i \cdot A_{i*}$ where A_{i*} is the i^{th} row of A

Equating the j^{th} component on both sides yields.

$$x_j = \sum_{i=1}^{m} \alpha_i \cdot a_{ij}$$

Since, $a_{ij} = [a_{ijL}, a_{ijU}]$

$$x_j = \sum_{i=1}^{m} \alpha_i \cdot [a_{ijL}, a_{ijU}]$$

$$= \sum_{i=1}^{m} [\alpha_i\, a_{ijL}, \alpha_i\, a_{ijU}] \qquad \text{(By Lemma (2.1.2(v)))}$$

$$= \left(\sum_{i=1}^{m} (\alpha_i . a_{ijL}) , \sum_{i=1}^{m} (\alpha_i . a_{ijU}) \right)$$

$$= [x_{jL}, x_{jU}].$$

x_{jL} is the j^{th} component of $x_L \in \mathcal{R}(A_L)$ and

x_{jU} is the j^{th} component of $x_U \in \mathcal{R}(A_U)$

Hence $x = [x_L, x_U]$

Therefore, $\mathcal{R}(A) = [\mathcal{R}(A_L), \mathcal{R}(A_U)]$

(ii) For $A = [A_L, A_U]$, the transpose of A is $A^T = [A_L{}^T, A_U{}^T]$

By using (i) we get,

$\mathcal{C}(A) = \mathcal{R}(A^T) = [\mathcal{R}(A_L{}^T), \mathcal{R}(A_U{}^T)] = [\mathcal{C}(A_L), \mathcal{C}(A_U)]$.

Hence the theorem

Theorem 2.1.7

For $A, B \in (IVFM)_{mn}$

(i) $\mathcal{R}(B) \subseteq \mathcal{R}(A) \Leftrightarrow B = XA$ for some $X \in (IVFM)_m$

(ii) $\mathcal{C}(B) \subseteq \mathcal{C}(A) \Leftrightarrow B = AY$ for some $Y \in (IVFM)_n$

Proof

Let $A = [A_L, A_U]$ and $B = [B_L, B_U]$

Since, $B = XA$, for some $X \in (IVFM)$,

Put $X = [X_L, X_U]$. Then by Lemma (2.1.2) (ii), $B_L = X_L A_L$ and $B_U = X_U A_U$

Hence, by Lemma (1.2.2), $\mathcal{R}(B_L) \subseteq \mathcal{R}(A_L)$ and $\mathcal{R}(B_U) \subseteq \mathcal{R}(A_U)$

By Theorem (2.1.6) (i), $\mathcal{R}(B) = [\mathcal{R}(B_L), \mathcal{R}(B_U)] \subseteq [\mathcal{R}(A_L), \mathcal{R}(A_U)] = \mathcal{R}(A)$.

Thus $\mathcal{R}(B) \subseteq \mathcal{R}(A)$

Conversely, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$

$\Rightarrow \mathcal{R}(B_L) \subseteq \mathcal{R}(A_L)$ and $\mathcal{R}(B_U) \subseteq \mathcal{R}(A_U)$ (By Theorem (2.1.6) (i))

$\Rightarrow B_L = YA_L$ and $B_U = ZA_U$ (By Lemma (1.2.2))

Then $B = [B_L, B_U]$

$= [YA_L, ZA_U]$

$= [Y,Z]\,[A_L, A_U]$ (By Lemma (2.1.2))

$= X[A_L, A_U]$, where $X = [Y, Z] \in (IVFM)_{mn}$

$= XA$

$B = XA$

ii) This can be proved along the same lines as that of (i) and hence omitted.

Theorem 2.1.8

For $A \in (IVFM)_{mn}$, $B \in (IVFM)_{np}$, the following hold.

$\mathcal{R}(AB) \subseteq \mathcal{R}(A)$ and $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$

Proof

Let $A = [A_L, A_U]$ and $B = [B_L, B_U]$

$A^T = [A_L^T, A_U^T]$ and $B^T = [B_L^T, B_U^T]$

Then by Lemma (2.1.2) (ii)

$AB = [A_L B_L, A_U B_U]$

By Theorem (2.1.6) (i)

$\mathcal{R}(AB) = \mathcal{R}([A_LB_L, A_UB_U]) = [\mathcal{R}(A_LB_L), \mathcal{R}(A_UB_U)] \subseteq [\mathcal{R}(A_L), \mathcal{R}(A_U)] = \mathcal{R}(A)$

(By Lemma (1.2.3))

Therefore, $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$

$\mathcal{C}(AB) = \mathcal{R}((AB)^T) = \mathcal{R}(B^T A^T) \subseteq \mathcal{R}(B^T) = \mathcal{C}(B)$

$\mathcal{C}(AB) = \mathcal{C}(B)$

Hence the Theorem.

Theorem 2.1.9

For *IVFM A, B* with $\mathcal{R}(A) = \mathcal{R}(B)$ or $\mathcal{C}(A) = \mathcal{C}(B)$, A is regular *IVFM* ⇔ *B* is regular *IVFM*.

Proof

Let *A* be a regular $(IVFM)_{mn}$ and $\mathcal{R}(A) = \mathcal{R}(B)$. Since $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ by Theorem (2.1.7) (i), $B = XA = XAA^{-}A = BA^{-}A$. $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ implies $YB = A$ for some $Y \in (IVFM)_m$.

Hence $B = BA^{-}A = BA^{-}(YB) = B(A^{-}Y)B = BZB$. Thus B is regular

Converse follows by interchanging A and B.

A is a regular *IVFM* ⇔ *B* is a *IVFM* under the condition $\mathcal{C}(A) = \mathcal{C}(B)$ can be proved along the same lines using Theorem (2.1.7) (ii) and hence omitted.

Hence the Theorem.

2.2 Invertible *IVFM*

In this section, by using the interval matrix representation of an *IVFM,* we exhibit that the class of all invertible $(IVFM)_n$ coincides with the invertible matrices in F_n, then, as an application, we show that the regularity of *IVFM* is preserved under congruence relation. In [41], the identity *IVFM* is defined as the *IVFM* in which all the diagonal entries are [*1, 1*] and all other entries are [*0, 0*]. It is denoted by I_n. By using the Definition (2.1.1), the representation (2.1.1) of the identity *IVFM* is $[I_n, I_n] = I_n$.

Definition 2.2.1

Let $A \in (IVFM)_n$, A is invertible $\Leftrightarrow AA^T = A^TA =$ Identity *IVFM* $= [I_n, I_n] = I_n$.

Definition 2.2.2

$P \in (IVFM)_n$ is a permutation matrix if each row and each column of P contains exactly one interval [1,1] and all the other entries are $[0,0]$.

Remark 2.2.3

We observe that $P \in (IVFM)_n$ is a permutation matrix $\Leftrightarrow PP^T = P^TP = I_n \Leftrightarrow P$ is invertible. Since no two permutation matrices are comparable, from (2.1.1) we have, $P_L = P_U = P \in F_n$.

Theorem 2.2.4

A is invertible $(IVFM)_n \Leftrightarrow A$ is invertible in F_n.

Proof

Let $A = [A_L, A_U]$, then $A^T = [A_L^T, A_U^T]$

A is invertible $\Leftrightarrow AA^T = A^TA =$ identity $(IVFM)_n$

$\Leftrightarrow [A_L, A_U]\,[A_L^T, A_U^T] = [I_n, I_n] = [A_L^T, A_U^T]\,[A_L, A_U]$ (By Definition (1.2.6))

$\Leftrightarrow [A_L A_L^T, A_U A_U^T] = [I_n, I_n] = [A_L^T A_L, A_U^T A_U]$ (By Lemma (2.1.2)(ii))

$\Leftrightarrow A_L A_L^T = I_n = A_L^T A_L$ and $A_U A_U^T = I_n = A_U^T A_U$

$\Leftrightarrow A_L$ and $A_U \in F_n$ are invertible.

$\Leftrightarrow A_L$ and A_U are Permutation matrices in F_n . (By Lemma(1.2.7)

Since $A_L \leq A_U$, by Remark (2.2.3), it follows that $A_L = A_U = A$.

Therefore, A is invertible $(IVFM)_n$ $\Leftrightarrow A$ is a Permutation matrix in F_n.

$\Leftrightarrow A$ is invertible in F_n.

Hence the Theorem.

Theorem 2.2.5

Let $A \in (IVFM)_{mn}$, $P \in (IVFM)_m$, $Q \in (IVFM)_n$. A is regular $\Leftrightarrow PAQ$ is regular for permutation matrices P and Q

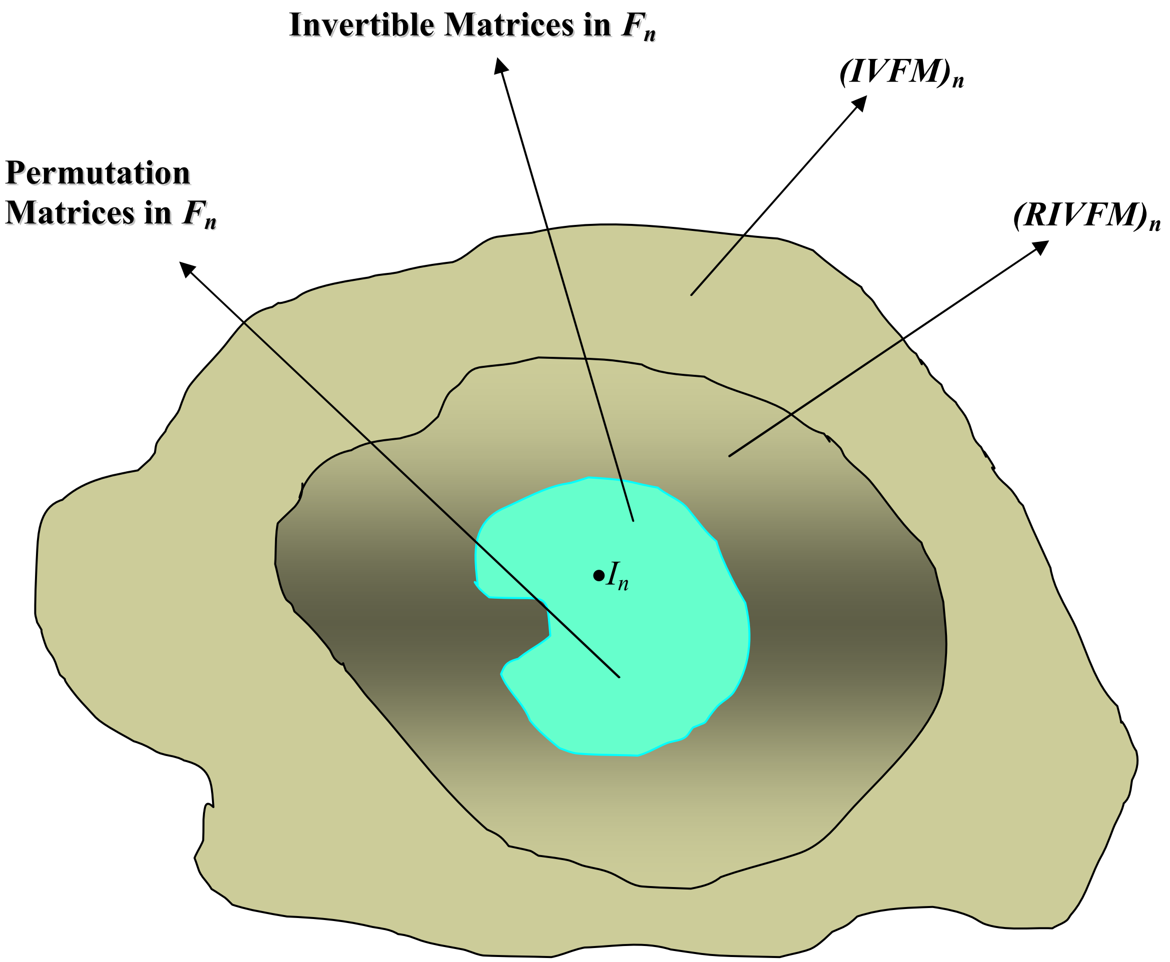
Proof

Let A be regular, then there exists $X \in (IVFM)$ such that $AXA = A$ since P, Q are Permutation matrices by Theorem (2.2.4) P and Q are invertible. It can be verified that $Q^T X P^T$ is a g-inverse of PAQ. Therefore PAQ is a regular $IVFM$.

Conversely, Let PAQ be regular then by the preceding part, $A = P^T(PAQ)Q^T$ is regular .

Hence the Theorem

From section (2.1) and section (2.2), we arrive at the following structure and inclusion relation among the sets of Interval valued fuzzy matrices of order n denoted as $(IVFM)_n$.



I_n is Identity matrix in F_n

$(IVFM)_n \supseteq$ Regular $(IVFM)_n \supseteq$ Invertible $(IVFM)_n \supseteq$ Permutation $(IVFM)_n$.

Invertible $(IVFM)_n$ = Invertible Matrices in F_n = Permutation Matrices in $F_{n.}$

(By Theorem (2.2.4))

Permutation $(IVFM)_n$ = Invertible Matrices in F_n (By Remark (2.2.3))

2.3 *g*- Inverses of *IVFM*

In this section, we have discussed the g-inverses of Interval Valued Fuzzy Matrices (*IVFM*) as a generalization of *g*- inverses of regular fuzzy matrices found in [19, 35]. The existence and construction of *g*-inverses, {*1, 2*} inverses, {*1, 3*} inverses and {*1, 4*} inverses of an Interval valued fuzzy matrix are determined in terms of the row and column spaces.

Definition 2.3.1

For $A \in (IVFM)_{mn}$ if there exists $X \in (IVFM)_{nm}$ such that

(1) $AXA = A$

(2) $XAX = X$

(3) $(AX)^T = (AX)$

(4) $(XA)^T = (XA)$, then *X is* called a *g*-inverse of *A*.

X is said to be a λ- inverse of *A* and $X \in A\{\lambda\}$ if *X* satisfies λ equation where λ is a subset of $\{1, 2, 3, 4\}$. $A\ \{\lambda\}$ denotes the set of all λ- inverses of *A*. In particular if $\lambda = \{1, 2, 3, 4\}$ then *X* unique and is called the Moore Penrose inverse of *A*, denoted as $A^{\dagger}$.

Remark 2.3.2

From Definition (2.3.1) of λ- inverses for $A \in (IVFM)$, by applying Lemma (2.1.2(ii)) for $A = [A_L, A_U]$ and $X = [X_L, X_U]$ it can be verified that the existence and construction of $\{\lambda\}$ inverses of $A \in (IVFM)_{mn}$ reduces to that of the $\{\lambda\}$- inverses of $A_L, A_U \in F_{mn}$.

Theorem 2.3.3

Let $A \in (IVFM)_{mn}$ and $X \in A\{1\}$, the following are equivalent

(i) $X = [X_L, X_U] \in A\{2\}$

(ii) $X_L \in A_L\{2\}$ and $X_U \in A_U\{2\}$

(iii) $\mathcal{R}(A_L X_L) = \mathcal{R}(X_L)$ and $\mathcal{R}(A_U X_U) = \mathcal{R}(X_U)$

(iv) $\mathcal{R}(AX) = \mathcal{R}(X)$

Proof

Since $A = [A_L, A_U]$ and $X = [X_L, X_U]$

(i) ⇒ (ii)

$X \in A\{2\} \Rightarrow XAX = X$, then by Lemma (2.1.2(ii)),

$$\Rightarrow [X_L, X_U]\,[A_L, A_U]\,[X_L, X_U] = [X_L, X_U]$$

$$\Rightarrow X_L A_L X_L = X_L, X_L \in A_L\{2\} \text{ and } X_U A_U X_U = X_U, X_U \in A_U\{2\}.$$

Thus **(ii)** holds

(ii) ⇒ (iii)

$$\Rightarrow X_L A_L X_L = X_L, X_L \in A_L\{2\} \text{ and } X_U A_U X_U = X_U, X_U \in A_U\{2\}.$$

$$\Rightarrow A_L \in X_L\{1\} \quad \text{and} \quad A_U \in X_U\{1\}$$

$$\Rightarrow \mathcal{R}(A_L X_L) = \mathcal{R}(X_L) \text{ and } \mathcal{R}(A_U X_U) = \mathcal{R}(X_U)$$

Thus **(iii)** holds

(iii) ⇒ (iv)

$$\Rightarrow \mathcal{R}(A_L X_L) = \mathcal{R}(X_L) \text{ and } \mathcal{R}(A_U X_U) = \mathcal{R}(X_U)$$

$$\Rightarrow \mathcal{R}(AX) = \mathcal{R}(X). \qquad \text{[By Theorem (2.1.6)]}$$

Thus **(iv)** holds

Conversely, let us prove **(iv) ⇒ (iii) ⇒ (ii) ⇒ (i)**

Let $\mathcal{R}(AX) = \mathcal{R}(X)$, then by Theorem (2.1.6), $\mathcal{R}(A_L X_L) = \mathcal{R}(X_L)$ and $\mathcal{R}(A_U X_U) = \mathcal{R}(X_U)$. Thus **(iii)** holds.

By Lemma (1.2.4) we have,

$\mathcal{R}(A_LX_L) = \mathcal{R}(X_L) \Rightarrow \mathcal{R}(X_L) \subseteq \mathcal{R}(A_LX_L)$ implies $X_L = Y_LA_LX_L$ for some $Y \in (IVFM)_m$.

$$\Rightarrow X_L(A_LX_L) = (Y_LA_LX_L)(A_LX_L)$$

$$\Rightarrow X_LA_LX_L = Y_L(A_LX_LA_L)X_L$$

$$= Y_LA_LX_L \qquad \text{[By Définition (2.3.1)]}$$

$$= X_L,\ X_L \in A\{2\}.$$

Similarly, $\mathcal{R}(A_UX_U) = \mathcal{R}(X_U) \Rightarrow X_U \in A\{2\}$. Thus **(ii)** holds.

Then by Theorem (2.1.6) we have, $\mathcal{R}(AX) = \mathcal{R}(X) \Rightarrow X \in A\{2\}$. Thus **(i)** holds.

Hence **(iv) ⇒ (iii) ⇒ (ii)⇒ (i)**

Hence the Theorem.

Remark 2.3.4

In the above Theorem (2.3.3), the condition X∈A{1} is essential. This is illustrated in the following example.

Example 2.3.5

Let $A = \begin{pmatrix} [0,1] & [1,1] \\ [1,1] & [0,0] \end{pmatrix}$, $X = \begin{pmatrix} [1,1] & [0,1] \\ [0,0] & [0,1] \end{pmatrix}$

Then by representation (1.2.1) we have, $A_L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A_U = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$,

$X_L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $X_U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$A_LX_LA_L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq A_L$ implies $X_L \notin A_L\{1\}$ and $A_UX_UA_U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq A_U$

.

implies $X_U \notin A_U\{1\}$

$$A_LX_L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_UX_U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

But $X_LA_LX_L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq X_L$ and $X_UA_UX_U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq X_U.$

Hence $X_L \notin A_L\{2\}$ and $X_U \notin A_U\{2\}$.

Then by Lemma (2.1.2(ii)) we have, $AXA \neq A,$ therefore $X \notin A\{1\}$

Here $\mathcal{R}(X_L) = \mathcal{R}(A_LX_L)$ and $\mathcal{R}(X_U) = \mathcal{R}(A_UX_U)$.

Therefore by Lemma (2.1.2(ii)), $\mathcal{R}(X) = \mathcal{R}(AX)$

But $XAX \neq X.$ Hence $X \notin A\{2\}$.

Theorem 2.3.6

For $A \in (IVFM)_{mn}$, A has a $\{1, 3\}$inverse if and only if A^TA is a regular $IVFM$ and $\mathcal{R}(A^TA) = \mathcal{R}(A)$.

Proof

Since A is regular, Theorem (2.1.3), A_L and A_U are regular. Let A has a $\{1, 3\}$ inverse X (say) then by Lemma (2.1.2(ii)), A_L has a $\{1, 3\}$ inverse X_L and A_U has a $\{1, 3\}$ inverse X_U.

Then $A_LX_LA_L = A_L$ and $(A_LX_L)^T = A_LX_L$

$$A_L^T(A_LX_LA_L) = A_L^TA_L$$

$$(A_L^TA_LX_L)A_L = A_L^TA_L$$

$\mathcal{R}(A_L{}^TA_L) \subseteq \mathcal{R}(A_L)$ [By Theorem (2.1.7)]

Similarly, $\mathcal{R}(A_U{}^TA_U) \subseteq \mathcal{R}(A_U)$

Therefore by Lemma (2.1.2(ii)) we have, $\mathcal{R}(A^TA) \subseteq \mathcal{R}(A)$

Also $(A_LX_L)^TA_L = A_LX_LA_L$

$\Rightarrow X_L{}^TA_L{}^TA_L = A_L$

$\Rightarrow X_L{}^T(A_L{}^TA_L) = A_L$

$\mathcal{R}(A_L) \subseteq \mathcal{R}(A_L{}^TA_L)$ [By Theorem (2.1.7)]

Similarly, $\mathcal{R}(A_U) \subseteq \mathcal{R}(A_U{}^TA_U)$

By Lemma (2.1.2 (ii)) we have, $\mathcal{R}(A) \subseteq \mathcal{R}(A^TA)$

Thus $\mathcal{R}(A) = \mathcal{R}(A^TA)$

Since $X \in A\{1\}$, $\mathcal{R}(A) = \mathcal{R}(XA)$

Hence $\mathcal{R}(A^TA) = \mathcal{R}(A) = \mathcal{R}(XA)$

Since $\mathcal{R}(A^TA) \supseteq \mathcal{R}(XA)$ [By Theorem (2.1.7)]

$YA^TA = XA$ let $Y = [Y_L, Y_U]$ then, $A_L{}^TA_L(Y_LA_L{}^TA_L) = A_L{}^TA_L(X_LA_L)$

$$(A_L{}^TA_L)Y_L(A_L{}^TA_L) = A_L{}^T(A_LX_LA_L)$$

$$= A_L{}^TA_L.$$

Similarly, $A_U{}^TA_U(Y_UA_U{}^TA_U) = A_U{}^TA_U$

By Lemma (2.1.2(ii)) we have, $A^TA(YA^TA) = A^TA$

Thus A^TA is a regular interval valued fuzzy matrix.

Conversely, let A^TA be a regular interval valued fuzzy matrix and $\mathcal{R}(A) = \mathcal{R}(A^TA)$.

By Lemma (1.2.5), A is regular *IVFM.*

Let us take $Y = (A^T)^-A^T \in (IVFM)$.

We claim that $Y \in A\{1, 3\}$.

$\mathcal{R}(A) = \mathcal{R}(A^TA)$ and A^TA is regular, by Lemma (1.2.4),

$A = A(A^TA)^-A^TA) = AYA$, $Y \in A\{1\}$ and since $\mathcal{R}(A) = \mathcal{R}(A^TA)$, $A = XA^TA$, by Theorem (2.1.3),

$A_L = X_LA_L{}^TA_L$ and $A_U = X_UA_U{}^TA_U$. Let $Y= [Y_L, Y_U]$.

Then, $A_LY_L = X_LA_L{}^TA_L\ (A_L{}^TA_L)^-A_L{}^T$

$= X_LA_L{}^TA_L\ (A_L{}^TA_L)^-A_L{}^TA_LX_L{}^T$

$= X_L\ (A_L{}^TA_L)(A_L{}^TA_L)^-\ (A_L{}^TA_L)X_L{}^T$

$= X_L\ (A_L{}^TA_LX_L{}^T)$

$= X_LA_L{}^T.$

Similarly, $A_UY_U = X_UA_U{}^T$.

Then by Lemma (2.1.2 (ii)) we have, $AY = XA^T$

$(A_LY_L)^T = (X_LA_L{}^T)^T$

$= A_LX_L{}^T$

$= X_LA_L{}^TA_LX_L{}^T$

$= X_LA_L{}^T = A_LY_L.$

Similarly, $(A_UY_U)^T = X_UA_U{}^T = A_UY_U$.

Then by Lemma (2.1.2 (ii)) we have, $(AY)^T = AY,\ Y\in A\{3\}$.

Since $\mathcal{R}(A) = \mathcal{R}(A^TA)$ by Lemma (1.2.4) and regularity of A^TA we get

$A = A(A^TA)^{-(}A^TA) = AYA,\ Y\in A\{1\}$

Thus A has a $\{1, 3\}$ inverse.

Hence the Theorem.

Theorem 2.3.7

For $A\in(IVFM)_{mn}$, A has $\{1, 4\}$ inverse if and only if AA^T is regular and $\mathcal{C}(AA^T) = \mathcal{C}(A)$.

Proof

This can be proved in the same manner as that of Theorem (2.3.6).

Corollary 2.3.8

Let $A\in(IVFM)_{mn}$ be a regular *IVFM* with A^TA is a regular *IVFM* and $\mathcal{R}(A^TA) = \mathcal{R}(A)$, then $Y = (A^TA)^-A^T\in A\{1, 2, 3\}$.

Proof

$Y \in A\{1, 3\}$ follows from Theorem (2.3.6), it is enough verify $Y = [Y_L, Y_U] \in A\{2\}$ that is, $Y_L A_L Y_L = Y_L$ and $Y_U A_U Y_U = Y_U$.

$$
\begin{aligned}
Y_L A_L Y_L &= Y_L (X_L^T A_L^T A_L) (A_L^T A_L)^- A_L^T \\
&= Y_L X_L^T (A_L^T A_L) (A_L^T A_L)^- (A_L^T A_L X_L) \\
&= Y_L X_L^T (A_L^T A_L) (A_L^T A_L)^- (A_L^T A_L) X_L \\
&= Y_L X_L^T A_L^T A_L X_L \\
&= Y_L A_L X_L \\
&= [(A_L^T A_L)^- A_L^T] A_L X_L \\
&= (A_L^T A_L)^- (A_L^T A_L X_L) \\
&= (A_L^T A_L)^- A_L^T \\
&= Y_L
\end{aligned}
$$

Similarly, $Y_U A_U Y_U = Y_U$. Then by Lemma (2.1.2 (ii)), $YAY = Y$.

Thus $Y \in A\{1,2, 3\}$.

Hence the Theorem.

Theorem 2.3.9

Let $A \in (IVFM)_{mn}$ be a regular *IVFM* with AA^T is a regular *IVFM* and $\mathcal{R}(A^T) = \mathcal{R}(AA^T)$ then $Z = A^T (AA^T)^- \in A\{1, 2, 4\}$.

Proof

Similar to the proof of Theorem (2.3.7) and corollary (2.3.8) hence omitted.

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3. INTERVAL VALUED BLOCK FUZZY MATRICES (*IVBFM*)

In this chapter, we discuss the regularity of Interval valued block fuzzy matrices *(IVBFM)* as a generalization of regularity of block fuzzy matrices studied in [33], and as an extension of regularity of *IVFM.* By introducing the concept of Schur complements for an Interval valued fuzzy matrix, equivalent condition for regularity of *IVBFM* are obtained and examples are provided (wherever necessary). We have discussed the generalized inverse formulae for an Interval valued Block Fuzzy Matrices. We derive equivalent conditions for the idempotency of a triangular Toeplitz Interval Valued Fuzzy Matrices *(IVFM)* of order up to 3, then we discuss the idempotency of a general triangular Toeplitz *IVFM* of order k.

3.1 Regular *IVBFM*

In this section, we introduce the concept of regularity for interval valued block fuzzy matrices *(IVBFM)* as a generalization of regularity of block fuzzy matrices in [33]. Necessary and sufficient conditions are obtained for the regularity of *IVBFM* in terms of the Schur complement of its regular diagonal blocks.

We derive a set of equivalent conditions for the regularity of *IVBFM*.

Throughout, we are concerned with an Interval valued block fuzzy matrix of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \rightarrow (3.1.1)$$

with A and D are regular *IVFM's*.

With respect to this partitioning a *Schur* complement of A in M is a Matrix of the form $M/A = D\text{-}CA^{-}B$ where A^{-} is some g–inverse of A. By M/A is a *IVFM,* we mean that $CA^{-}B$ is invariant and $D \geq CA^{-}B$. Therefore , M/A is Interval valued Fuzzy Matrix $\Leftrightarrow$ $CA^{-}B$ is invariant and $D + CA^{-}B = D$. Similarly $M/D = A\text{-}BD^{-}C$ → (3.1.2)

Let $A = [A_L, A_U]$, $B = [B_L, B_U]$, $C = [C_L ,C_U]$ and $D = [D_L, D_U]$. Let M of the form (3.1.1) M can be expressed as $M = [M_L ,M_U]$ where

$M_L = \begin{pmatrix} A_L & B_L \\ C_L & D_L \end{pmatrix}$ and $M_U = \begin{pmatrix} A_U & B_U \\ C_U & D_U \end{pmatrix}$ are block fuzzy matrices

Here we discuss the regularity of M for the cases *rank* M = *rank* A and *rank* $M \neq$ *rank* A

Since A and D are regular, by Theorem (2.1.3) A_L , A_U , D_L ,D_U are all regular Fuzzy Matrices.

By Lemma (1.2.10), $\rho_r (A_L) = \rho_c(A_L)$ and $\rho_r (A_U) = \rho_c(A_U)$, $\rho_r (D_L) = \rho_c (D_L)$ and $\rho_r (D_U) = \rho_c (D_U)$. Then by Theorem (2.1.6) we have,

$$\rho_r (A) = \rho_c(A) \text{ and } \rho_r (D) = \rho_c (D)$$

Lemma 3.1.1

For *IVFM* A, B, C if A is regular, $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{C}(B) \subseteq \mathcal{C}(A)$, then $CA^{-}B$ is invariant for all choices of g- inverses of A.

Proof

Since $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{C}(B) \subseteq \mathcal{C}(A)$, by Theorem (2.1.7) , $C = XA$ and $B = AY$, for some $X,Y \in (IVFM)$, now $CA^{-}B = (XA)A^{-} (AY) = X(AA^{-}A)Y=XAY$.

Thus $CA^{-}B$ is invariant of all choices of g-inverse of A.

Lemma 3.1.2

For *IVFM* A, B and C if A is regular then the following are equivalent:

(i) $\mathcal{R}(C) \subseteq \mathcal{R}(A)$

(ii) $\mathcal{R}(C_L) \subseteq \mathcal{R}(A_L)$, $\mathcal{R}(C_U) \subseteq \mathcal{R}(A_U)$

(iii) $C = CA^{-}A$ for all A^{-} of A

(iv) $C_L = C_L A_L^{-} A_L$ for all A_L^{-} of A_L and $C_U = C_U A_U^{-} A_U$ for all A_U^{-} of $A_{U.}$

Proof

Let $A = [A_L, A_U]$, $B = [B_L, B_U]$ and $C = [C_L, C_U]$

(i) ⇔ (ii)

Since $\mathcal{R}(C) = [\mathcal{R}(C_L), \mathcal{R}(C_U)]$ and $\mathcal{R}(A) = [\mathcal{R}(A_L), \mathcal{R}(A_U)]$

$\mathcal{R}(C) \subseteq \mathcal{R}(A) \Leftrightarrow \mathcal{R}(C_L) \subseteq \mathcal{R}(A_L)$ and $\mathcal{R}(C_U) \subseteq \mathcal{R}(A_U)$

Thus **(i)⇔(ii)** holds.

(ii)⇔(iv)

Since A is regular by Theorem (2.1.3), A_L, A_U are regular.

$\mathcal{R}(C_L) \subseteq \mathcal{R}(A_L) \Leftrightarrow C_L = X_L A_L$ (By Theorem (2.1.7))

$\Leftrightarrow C_L = X_L A_L A_L^{-} A_L$

$\Leftrightarrow C_L = C_L A_L^{-} A_L$ (By taking $X_L = C_L A_L^{-}$)

In the same manner $\mathcal{R}(C_U) \subseteq \mathcal{R}(A_U) \Leftrightarrow C_U = C_U A_U^{-} A_U$

Thus **(ii)⇔(iv)** holds.

(ii)⇔(iii)

Since A is regular by Theorem (2.1.3), A_L, A_U are regular. $\mathcal{R}(C_L) \subseteq \mathcal{R}(A_L)$ and $\mathcal{R}(C_U) \subseteq \mathcal{R}(A_U) \Leftrightarrow C_L = X_L A_L$ and $C_U = X_U A_U$ (By Theorem (2.1.7))

$\Leftrightarrow C_L = X_L A_L A_L^{-} A_L$ and $C_U = X_U A_U A_U^{-} A_U$

$\Leftrightarrow C_L = C_L A_L^{-} A_L$ and $C_U = C_U A_U^{-} A_U$ (By taking $X = CA^{-}$)

$\Leftrightarrow C = CA^{-}A$ (By Lemma (2.1.2)(ii)))

Thus **(ii)⇔(iii).**

Lemma 3.1.3

For *IVFM* A, B and C if A is regular then the following are equivalent:

(i) $\mathcal{C}(B) \subseteq \mathcal{C}(A)$

(ii) $\mathcal{C}(B_L) \subseteq \mathcal{C}(A_L)$, $\mathcal{C}(B_U) \subseteq \mathcal{C}(A_U)$

(iii) $B = AA^{-}B$ for all A^{-} of A

(iv) $B_L = A_L A_L^{-} B_L$ for all A_L^{-} of A_L and $B_U = A_U A_U^{-} B_U$ for all A_U^{-} of A_U

Proof

Since $\mathcal{C}(B) = \mathcal{R}(B^T)$, A is regular $\Leftrightarrow$ A^T is regular and $A^{-} \in A\{1\} \Leftrightarrow (A^{-})^T \in (A)^T\{1\}$.

Lemma (3.1.3) follows from Lemma (3.1.2).

Theorem 3.1.4

For *IVFM* A, B and C if A is regular then the following are equivalent:

(i) $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{C}(B) \subseteq \mathcal{C}(A)$

(ii) $C = CA^{-}A$ and $B = AA^{-}B$ for all A^{-} *of* A

(iii) $CA^{-}B$ is invariant, $C = C+CA^{-}A$ and $B = B+AA^{-}B$

Proof

Equivalence of (i) $\Leftrightarrow$ (ii) is precisely the equivalence of (i) and (iii) of Lemma (3.1.2) and Lemma (3.1.3).

(ii) $\Rightarrow$ (iii)

$C = CA^{-}A$ and $B = AA^{-}B$

$\Rightarrow C = C+CA^{-}A$ and $B = B+AA^{-}B$ (By Lemma (2.1.2)(ii)))

For any $A^{-} \in A\{1\}$, $CA^{-}B = (CXA)A^{-}(AYB)$, for some $X, Y \in A\{1\}$

$= C(XAY)B$

$= CZB \quad \text{where } Z = XAY \in A\{1\}$

Thus $CA^{-} B$ is invariant .

(iii) ⇒ (ii)

$B = B + AA^{-}B$ and $C = C + CA^{-}A \Rightarrow B \geq AA^{-}B$ and $C \geq CA^{-}A$.

Suppose $C > CA^{-}A$, then $CA^{-}B > C(A^{-}A\,A^{-})B = CXB$ where $X = A^{-}A\,A^{-}$ is a *g*-inverse of A. This contradicts the invariance of C. Hence $C = CA^{-}A$.

In the same manner, we can show that $B = AA^{-}B$.

Thus (ii) holds.

Remark 3.1.5

We note that condition (iii) in Theorem (3.1.4) cannot be relaxed. This can be seen by the following examples.

Example 3.1.6

Let us consider $A = \begin{pmatrix} [1,1] & [1,1] \\ [0.5,1] & [0,0] \end{pmatrix}$, $B = \begin{pmatrix} [0,0] & [0,0] \\ [0.1,1] & [0.1,1] \end{pmatrix}$ and $C = \begin{pmatrix} [0.1,1] & [0.1,1] \\ [0,0] & [0,0] \end{pmatrix}$

Then $A_L = \begin{pmatrix} 1 & 1 \\ 0.5 & 0 \end{pmatrix}$, $B_L = \begin{pmatrix} 0 & 0 \\ 0.1 & 0.1 \end{pmatrix}$, $C_L = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0 \end{pmatrix}$ and

$A_U = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $B_U = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $C_U = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

By using Algorithm 1 of [19] we have ,

For A_L, the set of all g- inverses of $A_L = \{ X/X = \begin{pmatrix} 0 & 1 \\ 1 & b \end{pmatrix},\ b \in \mathbf{F} \}$,

For A_U, the set of all *g*- inverse of $A_U = \{ X/X = \begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix},\ \alpha \in \mathbf{F} \}$;

$C_U A_U^{-} B_U = C_U$ for all *g*- inverses A_U^{-} of A_U.

Hence $C_U A_U^- B_U$ is invariant,
$B_U + A_U A_U^- B_U \neq B_U$, $C_U + C_U A_U^- A_U = C_U$. Similarly, $C_L A_L^- B_L$ is invariant, $B_L + A_L A_L^- B_L \neq B_L$, $C_L + C_L A_L^- A_L = C_L$. Hence CA^-B is invariant, $B + A\,A^-B \neq B$, $C + C\,A^-A = C$.

Therefore $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ holds and $\mathcal{C}(B) \subseteq \mathcal{C}(A)$ fails.

Example 3.1.7

Let us consider $A = [\,[0.5\,,0.6]]$, $B = [[0.3\,,0.5]\ \ [0.6\,,0.7]\,]$, $C = \begin{pmatrix} [0.6\,,0.7] \\ [0.5,\ 0.6] \end{pmatrix}$

Then,

$A_L = [0.5]$, $A_U = [0.6]$, $B_L = [0.3\ \ \ 0.6]$, $B_U = [0.5\ \ \ 0.7]$, $C_L = \begin{pmatrix} 0.6 \\ 0.5 \end{pmatrix}$ and $C_U = \begin{pmatrix} 0.7 \\ 0.6 \end{pmatrix}$

For $A_L = [0.5]$, each Matrix $A_L^- = [\alpha]$ for $\alpha \geq 0.5$ is a g-inverse of A_L and for $A_U = [0.6]$, each Matrix $A_U^- = [\alpha]$ for $\alpha \geq 0.6$ is a g - inverse of A_U. For $A_L^- = [0.5]$,

$C_L A_L^- B_L = \begin{pmatrix} 0.3 & 0.5 \\ 0.3 & 0.5 \end{pmatrix}$. For $A_L^- = [0.6]$, $C_L A_L^- B_L = \begin{pmatrix} 0.3 & 0.6 \\ 0.3 & 0.5 \end{pmatrix}$

Hence $C_L A_L^- B_L$ is not invariant. Similarly $C_U A_U^- B_U$ is not invariant. Therefore CA^-B is not invariant. Further,
$A_L A_L^- = A_L^- A_L = A_L$ for all g - inverse of A_L and $C_L A_L^- A_L = C_L A_L = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \neq C_L$;

$A_L A_L^- B_L = [0.3\ \ \ 0.5] \neq B_L$

Similarly $C_U A_U^- A_U \neq C_U$; $A_U A_U^- B_U \neq B_U$. Hence $CA^-A \neq C$; $A\,A^-B = B$.

Thus condition (ii) fails.

However $C_L + C_L A_L^- A_L = \begin{pmatrix} 0.6 \\ 0.5 \end{pmatrix} + \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.5 \end{pmatrix} = C_L$

$B_L + A_LA_L^-B_L = [0.3 \quad 0.6] + [0.3 \quad 0.5] = [0.3 \quad 0.6] = B_L$. Similarly, $C_U+C_UA_U^-A_U = C_U$, $B_U+A_UA_U^-B_U = B_U$. Therefore $C = C+CA^-A$ and $B = B+ AA^-B$.

Theorem 3.1.8

Let M be an block *IVFM* of the form (3.1.1) with $\mathcal{R}(C) \subseteq \mathcal{R}(A)$, $\mathcal{C}(B) \subseteq \mathcal{C}(A)$. Then the following are equivalent.

(i) $\mathcal{R}(B) \subseteq \mathcal{R}(D)$, $\mathcal{C}(C) \subseteq \mathcal{C}(D)$, the Schur complement M/A and M/D are Interval Valued Fuzzy Matrices.

(ii) M is regular, BD^-C is invariant and

$$m = \begin{pmatrix} A^-+A^-BD^-CA^- & A^-BD^- \\ D^-CA^- & D^- \end{pmatrix} \rightarrow (3.1.3)$$

is a g- inverse of M, for some g- inverse A^- of A and D^- of D.

Proof

Let $A = [A_L, A_U]$, $B = [B_L, B_U]$, $C = [C_L, C_U]$ and $D = [D_L, D_U]$

(i) ⇒ (ii)

Since A is regular by Theorem (2.1.3), A_L and A_U are regular. By Theorem (3.1.4), $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{C}(B) \subseteq \mathcal{C}(A) \Rightarrow C_L = C_LA_L^-A_L$ and $C_U = C_UA_U^-A_U$, $B_L = A_LA_L^-B_L$ and $B_U = A_UA_U^-B_U$ for each $A_L^- \in A_L\{1\}$, $A_U^- \in A_U\{1\}$. Since M/A is a Fuzzy matrix by (3.1.2), it follows that $D \geq CA^-B$ and $D = D+CA^-B$. Hence M can be expressed as $M = PEQ$ $M_L = P_LE_LQ_L$ and $M_U = P_UE_UQ_U$, where

$$P_L = \begin{pmatrix} I_L & 0 \\ C_LA_L^- & I_L \end{pmatrix}, \; E_L = \begin{pmatrix} A_L & 0 \\ 0 & D_L \end{pmatrix} \; and$$

$$Q_L = \begin{pmatrix} I_L & A_L^-B_L \\ 0 & I_L \end{pmatrix} \text{ and } P_U = \begin{pmatrix} I_U & 0 \\ C_UA_U^- & I_U \end{pmatrix}, \; E_U = \begin{pmatrix} A_U & 0 \\ 0 & D_U \end{pmatrix} \text{and } Q_U = \begin{pmatrix} I_U & A_U^-B_U \\ 0 & I_U \end{pmatrix}$$

Since $X+X = X$ under by Lemma (2.1.2(ii)), of fuzzy addition both P and Q idempotent, hence regular. On computation, we see that,

By (3.1.1) we have, $QE^-P = Q\begin{pmatrix} A^- & 0 \\ 0 & D^- \end{pmatrix} P =$ m defined in (3.1.3).

$$MmM = (PEQ)\,(QE^-P)\,(PEQ) = (PEQ)\,E^-\,(PEQ)$$

$$= \begin{pmatrix} A+BD^-C & B+BD^-D \\ C+DD^-C & D+CA^-B \end{pmatrix}$$

By Lemma (1.2.10) we have, M is regular and the matrix m in (3.1.3) is a g- inverse of M.

(ii) ⇒ (i)

Suppose the Matrix $m = M^-$ in (3.1.3) is a g- inverse of M, then comparing the corresponding blocks in MmM and M and using (3.1.1) and Lemma (2.1.2(ii)) we get,

$$A+BD^-C = A \qquad \rightarrow(3.1.4)$$

$$CA^-B+ CA^-B\,D^-CA^-B+D\,D^-CA^-B+ CA^-B\,D^-D+D = D \qquad \rightarrow(3.1.5)$$

$$C+ CA^-B\,D^-C+ D\,D^-C = C \qquad \rightarrow(3.1.6)$$

$$B+BD^-\,CA^-B + B\,D^-D = B \qquad \rightarrow(3.1.7)$$

By Theorem (3.1.4), $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{C}(B) \subseteq \mathcal{C}(A)$ implies CA^-B is invariant and $\mathcal{R}(B) \subseteq \mathcal{R}(D)$ and $\mathcal{C}(C) \subseteq \mathcal{C}(D)$ implies BD^-C is invariant.

Therefore M/A and M/D are Interval Valued Fuzzy Matrices.

Hence the Theorem.

Theorem 3.1.9

Let M be the form (3.1.1) with $\mathcal{R}(B) \subseteq \mathcal{R}(D)$ and $\mathcal{C}(C) \subseteq \mathcal{C}(D)$. Then the following are equivalent

(i) $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{C}(B) \subseteq \mathcal{C}(A)$ the *Schur* complement M/A and M/D are Interval Valued Fuzzy Matrices.

(ii) M is regular, CA^-B is invariant and

$m = \begin{pmatrix} A^- & A^-BD^- \\ D^-CA^- & D^-+D^-CA^-BD^- \end{pmatrix}$ is a g- inverse of M for some g-inverse A^- of A

and D^- of D. →(3.1.8)

Proof

This can be proved along the same lines as that of Theorem (3.1.8) and hence omitted.

Theorem 3.1.10

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $\mathcal{R}(C) \subseteq \mathcal{R}(A)$, $\mathcal{C}(B) \subseteq \mathcal{C}(A)$, $\mathcal{R}(B) \subseteq \mathcal{R}(D)$ and $\mathcal{C}(C) \subseteq \mathcal{C}(D)$.

If $M = [M_L, M_U]$ is regular then A and D are regular.

Proof

Let $A = [A_L, A_U]$, $B = [B_L, B_U]$, $C = [C_L, C_U]$ and $D = [D_L, D_U]$.

Let M be regular then by Theorem (2.1.3), M_L and M_U are regular and $m = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$ a g-inverse of $M \in IVFM$.

Then $MmM = M$. By comparing the corresponding diagonal blocks we get

$$AWA+BYA+AXC+BZC = A \qquad \rightarrow (3.1.9)$$

$$CWB+DYB+CXD+DZD = D \qquad \rightarrow (3.1.10)$$

By (3.1.1) and Theorem (2.1.7), under the given conditions on M, we have $C = UA$, $B = AV$, $C = DU_1$ and $B = V_1D$ for some fuzzy matrices U, V, U_1 and V_1. Substituting $B = AV$ and $C = UA$ in (3.1.9), we get $A(W+VY+XU+VZU)A = A$. Hence A is regular.

Similarly Substituting $B = V_1D$ and $C = DU_1$ in (3.1.10) yields that D is regular.

Hence the Theorem.

Theorem 3.1.11

Let M be as stated in Theorem (3.1.1) the following are equivalent.

(i) A and D are regular , their *Schu*r complement *M/A* and *M/D* are Interval Valued Fuzzy Matrices.

(ii) M is regular and M has a g- inverse $m = \begin{pmatrix} A^- & A^-BD^- \\ D^-CA^- & D^- \end{pmatrix}$ for some g- Inverse A^- of A and D^- of D. →(3.1.11)

Proof

This can be proved along the same lines as that of Theorem (3.1.8) and Theorem (3.1.10) and hence omitted.

3.2 g – Inverses of *IVBFM*

In this section, we shall see generalized inverse formulae for an Interval valued Block Fuzzy Matrix in the following.

Theorem 3.2.1

Let M be the Matrix as stated in Theorem (3.1.10). If A is $p \times q$ and D is $(l-p) \times (n-q)$ are regular matrices, with their *Schu*r complements *M/A* and *M/D* are Interval valued fuzzy matrices and

$$m = \begin{pmatrix} a & aBd \\ dCa & d \end{pmatrix} \quad \rightarrow (3.2.1)$$

Where a is $q \times p$ and d is $(n-p) \times (l-p)$ *IVFM* then we have the following:

(i) m is a semi-inverse of $M \Leftrightarrow a$ is a semi-inverse of A and d is a semi-inverse of D

(ii) m is a *{1,3}* inverse of $M \Leftrightarrow a$ is a *{1,3}*inverse of A, d is a *{1, 3}* inverse of D and $Bd = (Ca)^T$

(iii) m is a *{1,4}* inverse of $M \Leftrightarrow a$ is a*{1,4}* inverse of A, d is a *{1,4}* inverse of D and $aB = (dC)^T$

(iv) $m = M^{+} = M^{T} \Leftrightarrow a=A^{+}=A^{T}, d=D^{+}=D^{T}, BD^{T}=AC^{T}$ and $A^{T}B=C^{T}D$

(v) If A and d are square *IVFM*, then $M^{\#}$ exists $\Leftrightarrow A^{\#}$ and $D^{\#}$ exists, $BD=AB$ and $CD = CA$.

Proof

Let $A = [A_L, A_U]$, $B =[B_L, B_U]$, $C = [C_L, C_U]$ and $D = [D_L, D_U]$

By Theorem (3.1.11), M is regular and the matrix m in (3.2.1) which is precisely the matrix m defined in (3.1.11) for $a = A^{-}$ and $d = D^{-}$ is a g- inverse of M. Then the equivalence in (i), (ii) (iii) and (iv) are direct verification of m to satisfy the defining equations of generalized inverses. The proof for (v) runs as follows.

$M^{\#}$ exists $\Leftrightarrow Mm = mM$. Let $M = [M_L, M_U]$

$M^{\#}_L$ exists $\Leftrightarrow M_L m_L = m_L M_L$ and $M^{\#}_U$ exists $\Leftrightarrow M_U m_U = m_U M_U$

$$\Leftrightarrow \begin{pmatrix} A_L & B_L \\ C_L & D_L \end{pmatrix} \begin{pmatrix} A_L^{=} & A_L^{=} B_L D_L^{=} \\ D_L^{-} C_L A^{-}_L & D_L^{=} \end{pmatrix} = \begin{pmatrix} A_L^{=} & A_L^{=} B_L D_L^{=} \\ D_L^{=} C_L A^{=}_L & D_L^{=} \end{pmatrix} \begin{pmatrix} A_L & B_L \\ C_L & D_L \end{pmatrix}$$

$$M^{\#}_L \text{ exists} \Leftrightarrow \begin{pmatrix} A_L A_L^{\#} + B_L D_L^{=} C_L A^{=}_L & B_L D_L^{=} \\ C_L A_L^{=} & C_L A_L^{=} B_L D_L^{=} + D_L D_L^{=} \end{pmatrix} \rightarrow (3.2.2)$$

$$= \begin{pmatrix} A_L^{=} A_L + A_L^{=} B_L D_L^{=} C_L & A^{=}_L B_L \\ D^{=}_L C_L & D^{=}_L C_L A_L^{=} B_L + D_L^{=} D_L \end{pmatrix}$$

Similarly,

$$M_U^{\#} \text{ exists} \Leftrightarrow \begin{pmatrix} A_U A_U^{=} + B_U D_U^{=} C_U A_U^{=} & B_U D_U^{=} \\ C_U A_U^{=} & C_U A_U^{=} B_U D_U^{=} + D_U D_U^{=} \end{pmatrix} \rightarrow (3.2.3)$$

$$= \begin{pmatrix} A_U^{=}A_U + A_U^{=}B_U D_U^{=} C_U & A_U^{=}B_U \\ D_U^{=}C_U & D_U^{=} C_U A_U^{=} B_U + D_U^{=} D_U \end{pmatrix}$$

By Lemma (2.1.2(ii)) we have,

$$M^{\#} \text{ exists } \Leftrightarrow \begin{pmatrix} AA^{=}+BD^{=}CA^{=} & BD^{=} \\ CA^{=} & CA^{=}BD^{=}+DD^{=} \end{pmatrix} \quad \rightarrow (3.2.4)$$

$$\begin{pmatrix} A^{=}A+A^{=}BD^{=}C & A^{=}B \\ D^{=}C & D^{=}CA^{=}B+D^{=}D \end{pmatrix}$$

Since M/A and M/D Interval valued fuzzy matrices $A \geq BD^{-}C$ and $D \geq CA^{-}B$, $BD^{-}C$ and $CA^{-}B$ are invariant for all choice of g-inverses of A and D. By (3.1.7) we get $AA^{=}+BD^{=}CA^{=} = AA^{=}$, $CA^{=}+BD^{=}DD^{=} = DD^{=}$, $A^{=}A+A^{=}\,BD^{=}C^{=} = A^{=}A$ and $D^{=}D+D^{=}\,CA^{=}B = D^{=}D$.

Hence (3.2.4) reduces to

$$M^{\#} \text{ exists} \Leftrightarrow \begin{pmatrix} AA^{=} & BD^{=} \\ CA^{=} & DD^{=} \end{pmatrix} = \begin{pmatrix} A^{=}A & A^{=}B \\ D^{=}C & D^{=}D \end{pmatrix}$$

$\Leftrightarrow$ $A^{\#}$ and $D^{\#}$ exist , $BD^{=} = A^{=}B$ and $CA^{=} = D^{=}C$

$\Leftrightarrow$ $A^{\#}$ and $D^{\#}$ exist $ABD^{=}D = AA^{=}BD$ and $DCA^{=}A = DD^{=}CA$

$\Leftrightarrow$ $A^{\#}$ and $D^{\#}$ exist, $AB = BD$ *and* $CA = DC$ (By Theorem (3.1.4)

Hence the Theorem

Lemma 3.2.2

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $\rho_r(M) = \rho_r(A)$ and $\rho_C(M) = \rho_C(A)$.

Then there exist Interval valued fuzzy matrices X, Y such that $M = \begin{pmatrix} A & AY \\ XA & XAY \end{pmatrix}$

Proof

Let $A = [A_L, A_U]$, $B = [B_L, B_U]$, $C = [C_L, C_U]$, $D = [D_L, D_U]$ and $M = [M_L, M_U]$ Since $\rho_r\ (m) = \rho_r\ (A)$, the rows of C are linear combination of the rows of A, here $C = XA,$

$$[C_L, C_U] = [X_L, X_U]\ [A_L, A_U]$$

$$C_L = X_L A_L \text{ and } C_U = X_U A_U$$

$$C = XA \qquad \text{(By Lemma(2.1.2(ii)))}$$

For some Interval valued fuzzy matrices X. Similarly $\rho_c(m) = \rho_c(A)$ implies $B = AY$ for some interval valued fuzzy matrix Y and $D = XAY$.

Theorem 3.2.3

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $\rho_r(M) = \rho_r(A)$ and $\rho_c(M) = \rho_c(A)$. Then M is regular$\Leftrightarrow A$ is regular.

Proof

From Lemma (3.2.2), $C = XA$, $B = AY$ and $D = XAY$.

Hence, $M=ULV$ where $U = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}$, $L = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $V = \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}$

By (1.2.3), U and V are idempotent, hence regular. Suppose A is regular then L is regular $C = CA^{-}A$, $B = AA^{-}B$ and $D = CA^{-}B$.

We claim that $M = VL^{-}U$ is a g-inverse of M.

$MmM = (ULV)(VL^{-}U)(ULV) = ML^{-}M = M.$

Thus M is regular.

Conversely, let M be regular $M = [M_L, M_U]$

$$M = \begin{pmatrix} A & AY \\ XA & XAY \end{pmatrix}$$

$$M_L = \begin{pmatrix} A_L & A_L Y_L \\ X_L A_L & X_L A_L Y_L \end{pmatrix} \text{ and } M_U = \begin{pmatrix} A_U & A_U Y_U \\ X_U A_U & X_U A_U Y_U \end{pmatrix}$$

$$M_L = \begin{pmatrix} I_L \\ X_L \end{pmatrix} A_L \begin{pmatrix} I_L & Y_L \end{pmatrix} \text{ and } M_U = \begin{pmatrix} I_U \\ X_U \end{pmatrix} A_U \begin{pmatrix} I_U & Y_U \end{pmatrix} \quad \text{and}$$

$= U_L A_L V_L$ and $U_U A_U V_U$

$= UAV$ (by Lemma (2.1.2(ii))

Where $U = \begin{pmatrix} I \\ X \end{pmatrix}$, $V = \begin{pmatrix} I & Y \end{pmatrix} = UAV$

Clearly U and V are regular for $U^- = \begin{pmatrix} I & 0 \end{pmatrix}$ and $V^- = \begin{pmatrix} I \\ 0 \end{pmatrix}$ are g- inverses of U and V respectively.

Further, $U^- U = I$; $VV^- = I$ since M is regular for some M, $MmM = M$, by premultiplying with U^- and postmultiplying with V^-, we get $U^- UAVmUAVV^- = U^- UAVV^-$.

Hence, $A\ (VMU)\ A = A$.

Thus A is regular.

Hence the Theorem

3.3 Toeplitz *IVBFM*

In this section, first we derive equivalent conditions for the idempotency of a triangular Toeplitz Interval Valued Block Fuzzy Matrices *(IVBFM*) of order up to 3, then we discuss the idempotency of a general triangular Toeplitz *IVBFM* of order k of the form.

$$T_k = \begin{pmatrix} a & 0 & 0 & . & . & . & 0 \\ a_1 & a & 0 & . & . & . & 0 \\ a_2 & a_1 & a & . & . & . & 0 \\ . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & 0 \\ a_{k-2} & a_{k-3} & . & . & . & a & 0 \\ a_{k-1} & a_{k-2} & . & . & . & a_1 & a \end{pmatrix} \in (IVFM)_k \qquad \rightarrow (3.3.1)$$

where $a_i = [a_{iL}, a_{iU}]$, is a sub interval of $[0, 1]$ for each $i = 0$ to $k-1$

By representation (2.1.1) we have $T_k = [T_{kL}, T_{kU}]$.

Where,

$$T_{kL} = \begin{pmatrix} a_L & 0 & 0 & . & . & . & 0 \\ a_{1L} & a_L & 0 & . & . & . & 0 \\ a_{2L} & a_{1L} & a_L & . & . & . & 0 \\ . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & 0 \\ a_{(k-2)L} & a_{(k-3)L} & . & . & . & a_L & 0 \\ a_{(k-1)L} & a_{(k-2)L} & . & . & . & a_{1L} & a_L \end{pmatrix} \text{ and}$$

$$T_{kU} = \begin{pmatrix} a_U & 0 & 0 & . & . & . & 0 \\ a_{1U} & a_U & 0 & . & . & . & 0 \\ a_{2U} & a_{1U} & a_U & . & . & . & 0 \\ . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & 0 \\ a_{(k-2)U} & a_{(k-3)U} & . & . & . & a_U & 0 \\ a_{(k-1)U} & a_{(k-2)U} & . & . & . & a_{1U} & a_U \end{pmatrix} \qquad \rightarrow (3.3.2)$$

Lemma 3.3.1

Let $a = [a_L, a_U] \in (IVFM)$.

Then a is idempotent ⇔ a_L and $a_U \in F$ are idempotents.

Proof

Since $a = [a_L, a_U] \in (IVFM)$.

a is idempotent $\Leftrightarrow a^2=a$

$\Leftrightarrow [a_L^2, a_U^2] = [a_L, a_U]$ (By Lemma (2.1.2)(ii))

$\Leftrightarrow a_L^2 = a_L$ and $a_U^2 = a_U$

$\Leftrightarrow a_L$ and a_U are idempotents in F.

Thus, a is idempotent *IVFM* $\Leftrightarrow a_L$ and $a_U \in F$ are idempotents.

Theorem 3.3.2

(i) Each $a \in (IVFM)$ is idempotent as well as regular

(ii) $T_2 = \begin{pmatrix} a & 0 \\ a_1 & a \end{pmatrix} \in (IVFM)_2$ is idempotent $\Leftrightarrow a_1 \leq a$

(iii) $T_3 = \begin{pmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{pmatrix} \in (IVFM)_3$ is idempotent $\Leftrightarrow a_1 \leq a_2 \leq a$

Proof

(i) This follows from Lemma (2.1.3) and Lemma (3.3.1)

(ii) T_2 is idempotent then by (2.1.1), T_{2L} and T_{2U} are idempotents.

T_{2L} is idempotent $\Leftrightarrow T_{2L}^2 = T_{2L}$

$$\Leftrightarrow \begin{pmatrix} a_L & 0 \\ a_{1L}a_L & a_L \end{pmatrix} = \begin{pmatrix} a_L & 0 \\ a_{1L} & a_L \end{pmatrix}$$

$\Leftrightarrow a_{1L}a_L = a_L$

$\Leftrightarrow a_{1L} \leq a_L$ → (3.3.3)

Similarly, T_{2L} is idempotent $\Leftrightarrow a_{1U} \leq a_U$ → (3.3.4)

By (3.3.2) and (3.3.4) we have,

T_2 is idempotent $\Leftrightarrow T_{2L}$ and T_{2U} are idempotents

$\Leftrightarrow a_{1L} \leq a_L$ and $a_{1U} \leq a_U$

$\Leftrightarrow [a_{1L}, a_{1U}] \leq [a_L, a_U]$ (By(1.2.3))

$\Leftrightarrow a_1 \leq a$ (By(2.1.1)) *(iii)* Let as partition $T_3=\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in (IVFM)_3$

Where $A = \begin{pmatrix} a & 0 \\ a_1 & a \end{pmatrix} \in (IVFM)_2$, $D = [a]$, $C = [a_2, a] \in (IVFM)_{1X2}$

By (ii), A is idempotent $\Leftrightarrow a_1 \leq a$. D is idempotent

$$C_LA_L = C_L \Leftrightarrow [a_{2L} \quad a_{1L}] \begin{pmatrix} a_L & 0 \\ a_{1L} & a_L \end{pmatrix} = [a_{2L} \quad a_{1L}]$$

$$\Leftrightarrow a_{2L}\, a_L + a_{1L}\, a_L = a_{2L} \text{ and } a_{1L}a_L = a_{1L}$$

$$\Leftrightarrow (a_{2L} + a_{1L})\, a_L = a_{2L} \text{ and } a_{1L} \leq a_L \qquad \rightarrow (3.3.5)$$

Similarly, $$C_UA_U \quad \Leftrightarrow (a_{2U} + a_{1U})\, a_U = a_{2U} \text{ and } a_{1U} \leq a_U \qquad \rightarrow (3.3.6)$$

By (1.2.3), (3.3.5) and (3.3.6) we have,

$$CA = C \Leftrightarrow C_LA_L = C_L \text{ and } C_UA_U = C_U$$

$$\Leftrightarrow (a_2+a_1)a = a_2 \text{ and } a_1 \leq a$$

Similarly, $$DC = C \quad \Leftrightarrow D_LC_L = C_L \text{ and } D_UC_U = C_U$$

$$\Leftrightarrow a_2 \leq a \text{ and } a_1 \leq a$$

Thus

$$CA = DC = C \quad \Leftrightarrow a_1 \leq a,\ a_2 \leq a, \text{ and } (a_1+a_2)a = a_2$$

$$\Leftrightarrow a_1 \leq a_1 + a_2 \leq a, \quad a_2 \leq a_1 + a_2 \leq a \text{ and } (a_1+a_2)a = a_2$$

$$\Leftrightarrow a_1 \leq a,\ a_2 \leq a \text{ and } a_2 = (a_1+a_2)\, a = a_1+a_2$$

$$\Leftrightarrow a_1 \leq a,\ a_2 \leq a \text{ and } a_1 \leq a_2$$

$$\Leftrightarrow a_1 \leq a_2 \leq a.$$

By Lemma (1.2.11),

T_3 is an idempotent matrix $\Leftrightarrow$ the blocks A, D are idempotent matrices and $CA = DC = C \Leftrightarrow a_1 \leq a_2 \leq a$.

Hence the Theorem

Theorem 3.3.3

T_k is an idempotent *IVFM* if $a \geq a_{k-1} \geq a_{k-2} \geq \ldots \geq a_2 \geq a_1$.

Proof

By Theorem (3.3.2), the matrices T_1, T_2 and T_3 are idempotent under the condition $a \geq a_2 \geq a_1$. We prove T_4 is an idempotent *IVFM* under the condition $a \geq a_3 \geq a_2 \geq a_1$.

Let us partition $T_4 = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$, where $A = T_3$ is an idempotent *IVEM*. $D = [a]$ is idempotent *IVFM* and $C = [a_3 \ \ a_2 \ \ a_1] \in (IVFM)_{1X3}$. By Lemma (1.2.11) to prove that T_4 is idempotent *IVFM*, it is verify $CA = DC = C$ By (2.1.1), $C_LA_L = D_LC_L = C_L$ and $C_UA_U = D_UC_U = C_U$

$$C_LA_L = [a_{3L} \ \ a_{2L} \ \ a_{1L}] \begin{pmatrix} a_L & 0 & 0 \\ a_{1L} & a_L & 0 \\ a_{2L} & a_{1L} & a_L \end{pmatrix}$$

$= [(a_{3L}\, a_L) + (a_{1L}\, a_{2L}) \quad (a_{2L}\, a_L) + a_{1L} \quad a_{1L}\, a_L]$

$= [a_{3L} \ \ a_{2L} \ \ a_{1L}]$ Since $a_L \geq a_{3L} \geq a_{2L} \geq a_{1L}$

$= C_L$

Similarly, $C_UA_U = [a_{3U} \ \ a_{2U} \ \ a_{1U}] = C_U$

Then

$CA = [a_{3L} \ \ a_{2L} \ \ a_{1L}]$ and $[a_{3U} \ \ a_{2U} \ \ a_{1U}]$

$= [[a_{3L}, a_{3U}] \quad [a_{2L}, a_{2U}] \quad [a_{1L}, a_{1U}]]$

$= [a_3 \ \ a_2 \ \ a_1]$ since $a \geq a_3 \geq a_2 \geq a_1$ (by (2.1.1))

$= C$

Similarly, $DC = C$.

Hence T_4 is an idempotent *IVFM*. By induction on K, we can prove that T_K is an idempotent *IVFM* under the given condition.

Remark 3.3.4

If T_4 is an idempotent *IVFM*, then proceeding as in the proof of Theorem (3.3.2), we see that each $a_1, a_2, a_3 \leq a$, $a_1 \leq a_2$ but a_2 and a_3 need not be comparable. Hence we provide an example to show that the converse of Theorem (3.3.3) is not true.

Exemple 3.3.5

$$T_4 = \begin{pmatrix} [1,1] & 0 & 0 & 0 \\ [.3,.5] & [1,1] & 0 & 0 \\ [.5,.7] & [.3,.5] & [1,1] & 0 \\ [.4,.5] & [.5,.7] & [.3,.5] & [1,1] \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in (IVFM)_4$$

Where $A = \begin{pmatrix} [1,1] & 0 & 0 \\ [.3,.5] & [1,1] & 0 \\ [.5,.7] & [.3,.5] & [1,1] \end{pmatrix} \in (IVFM)_3$, $D = \begin{pmatrix} [1,1] \end{pmatrix}$

$C = \begin{pmatrix} [.4,.5] & [.5,.7] & [.3,.5] \end{pmatrix} \in (IVFM)_{1\text{x}3}$

By the representation (2.1.1) we have,

$A_L = \begin{pmatrix} 1 & 0 & 0 \\ .3 & 1 & 0 \\ .5 & .3 & 1 \end{pmatrix}$, $A_U = \begin{pmatrix} 1 & 0 & 0 \\ .5 & 1 & 0 \\ .7 & .5 & 1 \end{pmatrix}$, $D_L = [1]$, $D_U = [1]$,

$C_L = [.4 \quad .5 \quad .3]$ and $C_U = [.5 \quad .7 \quad .5]$

Since $1 \geq .5 \geq .3$ and $1 \geq .7 \geq .5$, by Theorem (3.3.2), A_L and A_U are idempotent.

$D_L = [1]$ and $D_U = [1]$ is idempotent.

$C_L A_L = [.4 \quad .5 \quad .3] \begin{pmatrix} 1 & 0 & 0 \\ .3 & 1 & 0 \\ .5 & .3 & 1 \end{pmatrix} = [.4 \quad .5 \quad .3] = C_L$ and

$$C_U A_U = [.5 \ \ .7 \ \ .5] \begin{pmatrix} 1 & 0 & 0 \\ .5 & 1 & 0 \\ .7 & .5 & 1 \end{pmatrix} = [.5 \ \ .7 \ \ .5] = C_U$$

$D_L C_L = 1[.4 \ \ .5 \ \ .3] = C_L$ and $D_U C_U = 1[.5 \ \ .7 \ \ .5] = C_U$

Then by Lemma (2.1.2(ii)) we have, $CA = C$ and $DC = C$. Hence by Lemma (1.2.7), T_4 is an idempotent *IVFM.*

However, the condition $a_2 \leq a_3$ fails.

Theorem 3.3.6

(i) Each $A \in (IVFM)$ is idempotent.

(ii) $T_2(a) = \begin{pmatrix} a_2 & a_1 \\ a_1 & a_0 \end{pmatrix} \in M_2(IVFM)$ is idempotent $\Leftrightarrow a_0 \geq a_1$

(iii) $T_3(a) = \begin{pmatrix} a_o & a_1 & a_2 \\ a_1 & a_o & a_1 \\ a_2 & a_1 & a_0 \end{pmatrix} \in M_3(IVFM)$ is idempotent $\Leftrightarrow a_0 \geq a_2 \geq a_1$

Proof

This can be proved along the same lines as that of Theorem (3.3.2) for Triangular Toeplitz *IVFM.*

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4. INTERVAL VALUED FUZZY RELATIONAL EQUATIONS

In this chapter, we discuss the consistency of the Interval valued *(I-V)* fuzzy relational equations as a generalization of that of fuzzy relational equations [38]. We have discussed the consistency of the Interval valued fuzzy relational equations and determine the complete set of solutions of $xA = b$ where A is an *IVFM* and b is an *IVF* vector, equivalent condition for the existence of Interval maximum solution of $xA = b$ for $A \in IVFM$ and its relation with the maximum solution of the system $xA^I = b^I$ is determined and illustrated with an example. We shall discuss the consistency of the interval valued fuzzy matrix equation $x \cdot M = b$, where let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in (IVFM)$, $x = [y\ z]$ and $b = [c\ d]$ be partitions of x and b respectively in conformity with that of M. A new algorithm is proposed to solve the interval valued fuzzy relation equation $P \circ Q = R$ with *max.min* composition and max product composition. The algorithm operates systematically on a matrix pattern to get all the solutions of P. An example is given to illustrate its effectiveness.

4.1 Consistency of I – V Fuzzy Relational Equations

In this section, we have discussed the consistency of the Interval valued fuzzy relational equations and the complete set of solutions of $xA = b$ where A is an Interval valued fuzzy matrix and b is an interval valued vector is determined. Equivalent condition for the existence of Interval maximum solution is obtained and illustrated with an example.

Let $xA = b$ be a system of Interval Valued Fuzzy Relational Equation, where A an Interval Valued Fuzzy Matrix$(IVFM)_{mn}$, b an Interval Valued Fuzzy Vector $(IVFV)_{1n}$ are given and to determine the unknown Interval Valued Fuzzy Vector x satisfying the equation $xA = b$ →(4.1.1)
where $A = [A_L, A_U]$ and $b = [b_L, b_U]$

Then by using Lemma (2.1.2 (ii)), $xA = b$ can be expressed as the following fuzzy relational equations

$x_L A_L = b_L$ →(4.1.2)
and $x_U A_U = b_U$ →(4.1.3)

Lemma 4.1.1

Let $xA = b$ be of the form (4.1.1). Then $\Omega(A, b) \neq \phi \Leftrightarrow \Omega(A_L, b_L) \neq \phi$ and $\Omega(A_U, b_U) \neq \phi$

Proof

Since $\Omega(A, b) \neq \phi$, there exists $x \in (IVFV)_n$ of the form $x = [x_L, x_U]$ satisfying the equation (4.1.1).
Then by using Lemma (2.1.2(ii)) we have, $x_L A_L = b_L$ and $x_U A_U = b_U$ which imply $x_L \in \Omega(A_L, b_L)$ and $x_U \in \Omega(A_U, b_U)$
Therefore $\Omega(A_L, b_L) \neq \phi$ and $\Omega(A_U, b_U) \neq \phi$.
Conversely, Let $\Omega(A_L, b_L) \neq \phi$ and $\Omega(A_U, b_U) \neq \phi$
We claim that $\Omega(A, b) \neq \phi$
For if we assume the contrary that is, $\Omega(A, b) = \phi$, then $xA \neq b$ for all $x \in (IVFV)_m$
Hence $[x_L, x_U]\,[A_L, A_U] \neq [b_L, b_U]$ for all $x_L, x_U \in F_m$
Then by Lemma (2.1.2(ii)) we have, $x_L A_L \neq b_L$ (or) $x_U A_U \neq b_U$ for all $x_L \in F_m$ and $x_U \in F_m$

Which implies $\Omega(A_L\ , b_L) = \phi$ (or) $\Omega(A_U, b_U) = \phi$
Which is a contradiction
Therefore $\Omega(A, b) \neq \phi$

Hence proved.

Lemma 4.1.2

Let $x\ A = b$ be an Interval valued fuzzy relational equation where $A = (a_{ij})$ and $b = (b_{ij})$ of the form (4.1.1). If $max_j\ (a_{ij}) < b_k$ for some $k\in N_n$, then $\Omega(A, b) = \phi$.

Proof

Since A is of the form (4.1.1), $A = [A_L,\ A_U]$ and $b = [b_L, b_U]$, by using (1.2.3), the given condition $max_j(a_{jk}) < b_k$ reduce to $max_j(a_{jkL}) < b_k$ for some $k\in N_n$ and $max_j(a_{jkU}) < b_k$ for some $k\in N_n$ where $b_k = [b_{kL}, b_{kU}]$. Then $min(x_{jL}.a_{jkL}) \leq a_{jkL} \leq max_j(a_{jkL}) < b_{kL}$. Hence $max\ min(x_{jL}, a_{jkL}) < b_{kL}$ for some $k \in N_n$ and by equation $max_{j\in Nm}\ min\ (a_{ijL}, b_{jkL}) = r_{ikL}$, for all $i\in N_s$ and $k\in N_n$, then by Lemma (1.2.14), no values $x_{jL}\in [0, 1]$ exist that satisfy the equation (4.1.2). Therefore $\Omega(A_L, b_L) = \phi$. Similarly $\Omega(A_U, b_U) = \phi$. Then by Lemma (4.1.1) we get, $\Omega(A, b) = \phi$.

Definition 4.1.3

Any element $\hat{x}$ *of* $\Omega(A, b)$ is said to be a maximum solution of $x\ A = b$, if $\hat{x} \geq x$ for all $x\in \Omega(A, b)$.

Remark 4.1.4

For $x, \hat{x}\in \Omega(A, b)$, if $x = [x_L, x_U]$ and $\hat{x} = [\hat{x}_L, \hat{x}_U]$. $\hat{x}$ is the maximum solution implies $\hat{x} \geq x$ for all $x\in \Omega(A, b)$. Hence $\hat{x}_L \geq x_L$ for all $x_L\in \Omega(A_L, b_L)$ and $\hat{x}_U \geq x_U$ for all $x_U \in \Omega(A_U, b_U)$. Then by (1.2.3), $\hat{x}_L$ is a maximum solution of $x_L A_L = b_L$ and $\hat{x}_U$ is a maximum solution of $x_U A_U = b_U$

Definition 4.1.5

Any element $\check{x}$ *of* $\Omega(A, b)$ is said to be a minimal solution of $xA = b$ if $\check{x} \leq x$ for all $x \in \Omega(A, b)$.

Remark 4.1.6

By the same argument as in Remark (4.1.4), if $\check{x} = [\check{x}_L, \check{x}_U]$ is a minimal solution of $xA = b$ then $\check{x}_L$ is a minimal solution of $x_L A_L = b_L$ and $\check{x}_U$ is a minimal solution of $x_U A_U = b_U$

Theorem 4.1.7

Let $xA = b$, $A \in (IVFM)_{mn}$, $b \in (IVFV)_{1n}$

$$\Omega(A,b) = \{x/x = [x_L, x_U] \text{ where } x_L \leq x_U,\ x_L \in \Omega(A_L, b_L) \text{ and } x_U \in \Omega(A_U, b_U)\} \rightarrow (4.1.4)$$

Proof

Let B denotes the set on the right side of the equation (4.1.4)

Let x be an arbitrary element of B, then $x = [x_L, x_U]$

where $x_L \leq x_U$, $x_L \in \Omega(A_L, b_L)$ and $x_U \in \Omega(A_U, b_U)$

Therefore $x_L A_L = b_L$ and $x_U A_U = b_U$

Now by using Lemma (2.1.2(ii)), we have

$$[x_L, x_U]\ [A_L, A_U] = [b_L, b_U]$$

$$x A = b$$

Which implies $x \in \Omega(A, b)$

Since x is arbitrary element of B, it follows that $B \subseteq \Omega(A,b)$

Next to prove $\Omega(A,b) \subseteq B$, let $x \in \Omega(A,b)$. Then $x \in (IVFV)_{1m}$ satisfying the equation $xA = b$, Let $x = [x_L, x_U]$ with $x_L \leq x_U$ Then by using Lemma (2.1.2(iii)) we have,

$[x_L, x_U]\ [A_L, A_U] = [b_L, b_U]$

Hence $x_L \leq x_U$ Such that, $x_L A_L = b_L$ and $x_U A_U = b_U$

$\Rightarrow x \in B$

Therefore $\Omega(A,b) \subseteq B$

Hence proved

Corollary 4.1.8

If $\Omega(A,b) \neq \phi$ then $xA = b$ has a unique maximum solution $\Leftrightarrow \hat{x}_L \leq \hat{x}_U$ where $\hat{x}_L$ is the unique maximum solution of (4.1.2) and $\hat{x}_U$ is the unique maximum solution of (4.1.3).

Proof

Since $\Omega(A,b) \neq \phi$, by Lemma (4.1.1), $\Omega(A_L, b_L) \neq \phi$ and $\Omega(A_U, b_U) \neq \phi$. If $\hat{x}$ is the unique maximum solution then by Theorem (4.1.7), $\hat{x} = [\hat{x}_L, \hat{x}_U]$ and by Remark (4.1.4), $\hat{x}_L$ is the unique maximum solution of (4.1.2) and $\hat{x}_U$ is the unique maximum solution of (4.1.3).

Conversely

If $\hat{x}_L \leq \hat{x}_U$ then define $\hat{x} = [\hat{x}_L, \hat{x}_U]$

We claim that $\hat{x}$ is maximum solution for $xA = b$

For if not then by Remark (4.1.4), then either $\hat{x}_L$ is not a maximal solution of (4.1.2) (or) $\hat{x}_U$ is not a maximal solution of (4.1.3)

Which is a contradiction

Therefore $\hat{x} = [\hat{x}_L, \hat{x}_U]$ is the maximum solution for $x A = b$.

Remark 4.1.9

By Lemma (4.1.1) and Theorem (4.1.7), if $\hat{x} = [\hat{x}_L , \hat{x}_U]$ is the maximum solution of $x\,A = b$, then $\hat{x}_L$ is a maximal solution of (4.1.2) and $\hat{x}_U$ is a maximal solution of (4.1.3) and if $\hat{x}_L \not\leq \hat{x}_U$, then $xA = b$ has no maximum solution. This is illustrated in the following:

Illustration 4.1.10

$$\text{Let } A = \begin{pmatrix} [0.3,0.7] & [0.5, 0.6] \\ [0.4,0.9] & [0.1, 0.7] \\ [0.2,0.5] & [0.8,1.0] \end{pmatrix}$$

$b = ([0.4, 0.8] \quad [0.5,0.7])$, then by representation (4.1.1) we have,

$$A_L = \begin{pmatrix} 0.3 & 0.5 \\ 0.4 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} \qquad A_U = \begin{pmatrix} 0.7 & 0.6 \\ 0.9 & 0.7 \\ 0.5 & 1.0 \end{pmatrix}$$

$$b_L = [0.4 \quad 0.5] \quad \text{and} \quad b_U = [\,0.8 \quad 0.7]$$

First let us find out the maximum solution $\hat{x}_L$ of the equation $x_L\ A_L = b_L$ by applying (2.1.1) for the matrix A_L.

$$\hat{x}_{1L} = \min_{j\in J} (\, a_{1jL}, b_{jL}) = \min\{1, 1\} = 1$$

$$\hat{x}_{2L} = \min_{j\in J} (\, a_{2jL}, b_{jL}) = \min\{1, 1\} = 1$$

$$\hat{x}_{3L} = \min_{j \in J} (a_{3jL}, b_{jL}) = \min\{1, 0.5\} = 0.5$$

and

$$[1 \quad 1 \quad 0.5] \begin{pmatrix} 0.3 & 0.5 \\ 0.4 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} = [0.4 \quad 0.5]$$

Thus $\hat{x}_L A_L = b_L$ and $\Omega\,(A_L, b_L) \neq \phi$

$\hat{x}_L = [1, 1, 0.5]$ is the maximum solution of $x_L A_L = b_L$

$\hat{x}_U = [1, 0.8, 0.7]$ is the maximum solution of $x_U A_U = b_U$

Here $\hat{x}_L \nleq \hat{x}_U$. Therefore $[\hat{x}_L, \hat{x}_U]$ is not an interval valued fuzzy vector and it is not a solution for the system $x A = b$. Hence $xA = b$ has no maximum solution.

We shall see an extension of the results in [6] related to the consistency of system of the fuzzy relational equation in terms of regularity of the coefficient matrix in the following:

Theorem 4.1.11

For $A \in (IVFM)_{mn}$, $b \in (IVFV)_{1n}$. If A is regular then $xA = b$ is consistent and $x = bX$ is a solution for some g–inverse X of A

Proof

If A is regular then by Lemma (2.1.3), both A_L and A_U are regular by Remark (2.1.9), $X = [X_L, X_U]$ is a g–inverse of A, for some $X_L \in A_L\{1\}$ and $X_U \in A_U\{1\}$

Since $xA = b$ is consistent and $\Omega\,(A, b) \neq \phi$. By using (4.1.1), $xA = b$ reduce to the following fuzzy relational equations

$x_L A_L = b_L$ and $x_U A_U = b_U$

Since A_L is regular the equation $x_L A_L = b_L$ consistent and $xA_L X_L A_L = b_L$ for some $X_L \in A_L\{1\} \Rightarrow b_L X_L (A_L) = b_L$ and

$b_L X_L$ is a solution for the equation (4.1.2). → (4.1.5)

Similarly (4.1.3) is consistent and $b_U X_U$ is a solution → (4.1.6)

By using (4.1.5) and (4.1.6) we have $b = [b_L, b_U]$

$$= [b_L X_L A_L,\ b_U X_U A_U]$$

$$= b[X_L, X_U]\,[A_L, A_U]$$

$$= bXA \qquad \text{(By Lemma (2.1.2(ii)))}$$

Hence bX is a solution for the equation $xA = b$. Thus whenever A is regular the equation $xA = b$ is consistent.

Hence the Theorem.

Next we shall introduce the upper and lower solutions for the general equation $x A = b$ where $A = [A_L, A_U] \in (IVFM)_{mn}$, $b = [b_L, b_U] \in (IVFV)_{1n}$.

Definition 4.1.12

The upper solutions for the equation $xA = b$ is the set of all solutions satisfying $xA \geq b$ and lower solutions for the equation $xA = b$ is the set of all solutions satisfying $xA \leq b$.

Let $\Omega_U(A, b)$ and $\Omega_L(A, b)$ be the set of all upper solutions and lower solutions respectively.

Thus $\Omega(A, b) = \Omega_U(A, b) \cap \Omega_L(A, b)$

Theorem 4.1.13

Let $xA^I = b^I$, where $A^I = \{ A' / A_L \leq A' \leq A_U\}$ and $b^I = \{ b'/ b_L \leq b' \leq b_U\}$. Then $y \in \Omega(A^I, b^I) \Leftrightarrow y \in \Omega_U(A', b_L) \cap \Omega_L(A', b_U)$

Proof

Since, $A_L \le A_U$, consider the set of solution $A' \in F_{mn}$ such that,

$$A_L \le A' \le A_U$$

$$\Rightarrow x\,A_L \le xA' \le xA_U$$

$$\Rightarrow b_L \le xA' \le b_U \qquad \text{(By equations (4.1.2) and (4.1.3)}$$

$$\Rightarrow b_L \le y \le b_U$$

Thus, $y = xA' \ge b_L$ and $y = xA' \le b_U$

Therefore $y \in \Omega_U(A', b_L) \cap \Omega_L(A', b_U)$

By Lemma (1.2.18) we have,

$$y \in \Omega(A^I, b^I) \Leftrightarrow y \in \Omega_U(A', b_L) \cap \Omega_L(A', b_U).$$

Then the following solution sets are identical

(i) $\Omega\,(A^I, b^I)$

(ii) $\bigcup_{A' \in A^I} (\Omega_U(A', b_L) \cap \Omega_L(A', b_U)$

(iii) $\bigcup_{\substack{A' \in A^I \\ b' \in b^I}} \Omega\,(A', b')$.

Theorem 4.1.14

The maximum solution of $xA = b$ where $A \in (IVFM)_{mn}$ coincides with x^* in the maximum element $y^* = x^*A_U$ in $\Omega(A^I, b^I)$, where A_U is the upper limit of A.

Proof

By Lemma (1.2.19), the maximum element $y^* \in \Omega(A^I\ b^I)$ is computed as $y^* = x^*A_U$.

$$x^* = A_{U} \cdot \circ b_U = \left(\begin{array}{l} \min\ (a_{ijL} \circ b_{jU}) \\ j \in J \end{array} \right)_{i \in I}$$

Where $$a_{ijU} \circ b_{jU} = \begin{cases} 1, & \text{if } a_{ijU} \leq b_{jU} \\ b_{jU}, & \text{if } a_{ijU} \leq b_{jU} \end{cases}$$

Thus by (2.1.1) applied to the matrix A_U, we see that x^* coincides with the maximum solution $\hat{x}_U$, of the equation $x_U A_U = b_U$. Then by Theorem (4.1.7), we note that the upper limit of the maximum solution of $xA = b$ where $A \in (IVFM)_{mn}$ coincides with x^* in the maximum element $y^* = x^* A_U$ in $\Omega(A^I, b^I)$.

4.2 Interval Valued Block Fuzzy Relational Equations

In this section, we shall discuss the consistency of the interval valued fuzzy matrix equation $x.M = b$. Throughout this section, let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in (IVFM)$, $x = [y\ z]$ and $b = [c\ d]$ be partitions of x and b respectively in conformity with that of M.

Theorem 4.2.1

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be of the form (3.1.1), with A and D are regular. M/A and M/D exist. $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(B) \subseteq \mathcal{R}(D)$. Then $x \cdot M = b$ is solvable if and only if $y \cdot A = c$ and $z \cdot D = d$ are solvable, $c \geq dD^{-}C$ and $d \geq cA^{-}B$.

Proof

Let $A = [A_L, A_U]$, $B = [B_L, B_U]$, $C = [C_L, C_U]$ and $D = [D_L, D_U]$.

Since A is regular by Theorem (2.1.3), A_L and A_U are regular.

Suppose $x{\cdot}M = b$ is solvable and $\alpha = [\beta\gamma]$ is a solution.

Then by Lemma (2.1.2(ii)), $x_L{\cdot}M_L= b_L$ is solvable and $x_U{\cdot}M_U = b_U$ is solvable and $\alpha = [\beta\gamma]$ is solution. Then we get, $\beta A_L+ \gamma C_L = c_L$, $\beta B_L+ \gamma D_L = d_L$ and

$\beta A_U + \gamma C_U = c_U$, $\beta B_U + \gamma D_U = D_U$ → (4.2.1)

Since $\mathcal{R}(C_L) \subseteq \mathcal{R}(A_L)$ and $\mathcal{R}(B_L) \subseteq \mathcal{R}(D_L)$, by Theorem (3.1.4), $C_L = C_LA_L^-A_L$ and $BL = B_LD_L^-D_L$. Substituting for C_L and B_L in (4.2.1) we get $(\beta+\gamma C_LA_L^-)A_L = c_L$ and $(\beta B_LD_L^- + \gamma)D_L = d_L$.

Thus $y_L{\cdot}A_L= c_L$ and $z_L{\cdot}D_L=d_L$ are solvable.

Since A_L and B_L are regular, the solutions will be of the form $y_L= c_LA_L^-$ and $z_L= d_LD_L^-$.

Hence $c_LA_L^- = \beta + \gamma C_LA_L^-$ and $d_LD_L^- = \beta B_LD_L^- + \gamma$.

Similarly, $c_UA_U^- = \beta + \gamma C_UA_U^-$ and $d_UD_U^- = \beta B_UD_U^- + \gamma$.

By Lemma (2.1.2) we have, $cA^- = \beta + \gamma CA^-$ and $dD^- = \beta BD^- + \gamma$

$cA^-B = \beta B + \gamma CA^-$ and $dD^-C = \beta BD^-C + \gamma C$ → (4.2.2)

Since M/A and M/D exist by (3.1.2), $A+BD^-C =A$ and $D+CA^-B$ substituting for A and D in (4.2.1) using (4.2.2) we get $c = \beta A + \gamma C = \beta A + \beta BD^-C + \gamma C = \beta A + dD^-C$, $d = \beta B +\gamma D = \beta B + \gamma D + \gamma CA^-B = \gamma D + cA^-B$.

By fuzzy addition it follows that $c \geq dD^-C$ and $d \geq cA^-B$ as required.

Conversely, suppose $y{\cdot}A = c$ and $z{\cdot}C = d$ are solvable, then by Lemma (2.1.2), $y_L{\cdot}A_L = c_L$ and $z_L{\cdot}C_L = d_L$ are solvable and $y_U{\cdot}A_U = c_U$ and $z_U{\cdot}C_U = d_U$ are solvable, then $y_L = c_LA_L^-$ and $z_L = d_LD_L^-$ and $y_U = c_UA_U^-$ and $z_U = d_UD_U^-$.

Therefore $C_LA_L^-A_L = c_L$, $d_LD_L^-D_L = d_L$ and $C_UA_U^-A_U = c_U$ and $d_UD_U^-D_U = d_U$

Hence by Lemma (2.1.2(ii)) we have, $CA^-A = c$ and $dD^-D = d$.

From the given conditions $c \geq dD^-C$ and $d \geq cA^-B$ we get, $c + dD^-C = c$ and $d + cA^-B =d$.

Now $[\,cA^{-}\quad dD^{-}]\begin{pmatrix} A & B \\ C & D \end{pmatrix} = [cA^{-}A + dD^{-}C \qquad cA^{-}B + dD^{-}D]$

$$= [c + dD^{-}C \qquad d + cA^{-}B]$$

$$= [c \quad d]$$

Thus $x{\cdot}M = b$ is solvable.

Hence the Theorem.

Remark 4.2.2

In particular, for $B = 0$, the above Theorem reduces to the following.

Corollary 4.2.3

For the Matrix $M = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ be of the form (3.1.1), such that $\mathcal{R}(C) \subseteq \mathcal{R}(A)$, the blocks A and D are regular matrices, the following statements are equivalent.

(i) $x{\cdot}M = b$ solvable.

(ii) $y{\cdot}A = c,\ z{\cdot}D = d$ are solvable and $c \geq dD^{-}C$.

Proof

Let $A = [A_L, A_U]$, $C = [C_L, C_U]$ and $D = [D_L, D_U]$.

(i)⇒ (ii) Since M is regular by Theorem (3.1.10), A and D are regular.

Suppose $x{\cdot}M = b$ is solvable then by Lemma (2.1.2), $x_L{\cdot}M = b_L$ and $x_U{\cdot}M = b_U$ are solvable, let $\alpha = [\beta\gamma]$ be a solution. On substitution gives $\beta{\cdot}A_L + \gamma{\cdot}C_L = c_L$ and $\gamma D_L = d_L$.

Since $R(C_L) \subseteq R(A_L)$ by using $C_L = C_L A_L^{-} A_L$ we get, $(\beta + \gamma C_L A_L^{-})A_L = c_L$ and $\gamma{\cdot}D_L = d_L$. Therefore $y_L{\cdot}A_L = c_L$ and $z_L{\cdot}D_L = d_L$ are both solvable with $y_L = \beta + \gamma C_L A_L^{-}$ is solution of $y_L{\cdot}A_L = c_L$ and $z_L = \gamma$ is a solution of $z_L{\cdot}D_L = d_L$.

Since D is regular then by Lemma (2.1.3), D_L and D_U are regular. Then $\gamma = d_L D_L^-$ is a solution of $z_L \cdot D_L = d_L$. Now, $\gamma C_L = d_L D_L^- C_L$.

From $\beta A_L + \gamma C_L = c_L$ by fuzzy addition, we get $c_L \geq \gamma C_L = d_L D_L^- C_L$.

Similarly, $y_U \cdot A_U = c_U$, $z_U \cdot D_U = d_U$ are solvable and $c_U \geq d_U D_U^- C_U$.

Hence by Lemma (2.1.2(ii)) we have, $y \cdot A = c$, $z \cdot D = d$ are solvable and $c \geq dD^- C$.

(ii)⇒ (i) Suppose $y \cdot A = c$, $z \cdot D = d$ are solvable then by Lemma (2.1.2), $y_L \cdot A_L = c_L$, $z_L \cdot D_L = d_L$ are solvable and $y_U \cdot A_U = c_U$, $z_U \cdot D_U = d_U$ are solvable.

Since both A_L and D_L are regular matrices. $y_L = c_L A_L^-$ and $z_L = dD_L^-$ are respectively the solutions of the equation $y_L \cdot A_L = c_L$ and $z_L \cdot D_L = d_L$.

Hence $c_L A_L^- A_L = c_L$ and $d_L D_L^- D_L = d_L$.

Similarly, $c_U A_U^- A_U = c_U$ and $d_U D_U^- D_U = d_U$.

By Lemma (2.1.2(ii)) we have, $cA^- A = c$ and $dD^- D = d$ by fuzzy addition $c \geq dD^- C$ implies $c + dD^- C = c$.

$$[c\,A^- d\,D^-]\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = [c\,A^- A + d\,D^- C \qquad d\,D^- D]$$

$$= [c + d\,D^- C \qquad d]$$

$$= [c \qquad d] = b.$$

Thus $[cA^- \;\; dD^-]$ is a solution of the equation $x \cdot M = b$.

Hence $x \cdot M = b$ is solvable.

Remark 4.2.4

In corollary (4.2.3), the conditions on M need not imply M is regular. Hence we have determined equivalent conditions for consistency of $x \cdot M = b$ without M being regular.

Corollary 4.2.5

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be of the form (3.1.1) with $\rho_r(M) = \rho_r(A)$, $\rho_c(M) = \rho_c(A)$ and A is regular. Then $x\cdot M = b$ is solvable if and only if $y\cdot A = c$ is solvable and $cA^{-}B = d$.

Proof

This follows from Theorem (4.1.1).

4.3 New Algorithms for solving *I –V* Fuzzy relational Equations

In this section, new algorithm is proposed to solve the Interval valued *(I – V)* fuzzy relational equation $P\cdot Q = R$ with *max.min* composition and max product composition. The algorithm operates systematically and graphically on a matrix pattern to get all the interval solution of P. An example is given to illustrate its effectiveness.

Let interval valued Fuzzy matrix $Q = [q_{jk}]_{mxn}$ be called the state – matrix and Interval valued fuzzy vector $r = [r_k]$ be called the output vector with $q_{jk} \in [0, 1]$, $r_k = [0,1]$ for all $j \in J$ and $k \in K$, where $j = [\ 1,2,....m]$ and $K = \{1,2,.....,n\}$, $m,n \in N$. The problem is to determine all vectors $p \in P = \{\ p = [p_j]_{1xm}\ / p_j \in [\ 0,\ 1\ ]\}$ satisfying Interval valued fuzzy relational equation.

$p\cdot Q = r$ → 4.3.1

Then by using Lemma (2.1.2) $p\cdot Q = r$ can be expressed as the following fuzzy relational equations

$p_L.\, Q_L = r_L$ → 4.3.2

and $p_U\, Q_U = r_U$ → 4.3.3

Where denotes the *max.min* composition with $\max_{j \in J} \min(p_j, q_{jk}) = r_k$ or *max – product* composition with $\max_{j \in J} (p_j \cdot q_{jk}) = r_k$ for all $k \in K$.

The following algorithm and Theorems are the generalization of results found in [25].

Algorithms 4.3.1

Following are the algorithms for solving equation (4.3.1) with *max.min* (or *max – Product*) composition:

Step 1 : Check the existence of the solution [By Section 4.1.1]

Step 2: Rank the elements of r with decreasing order and find the maximum solution $\bar{p}$ [By Section 4.1.1]

Step3: Build $M=[m_{jk}]$, $j=1,2,\ldots m$: $k=1,2,\ldots n$. were $m_{jk} \triangleq (\bar{p}_j, q_{jk})$.
Let $M=[M_L, M_U]$. Therefore $m_{jk} = [m_{jkL}, m_{jkU}]$
Hence $M_L = [m_{jkL}]$ and $M_U = [m_{jkU}]$
This interval matrix M is called the matrix pattern.

Step 4: Interval matrix m_{jk}, which satisfies $\min(\bar{p}_j, q_{jk}) = r_k$ (or $p_j.\ q_{jk} = r_k$), and then let the marked m_{jk} be denoted by $\overline{m}_{jk}$.

Step 5: If k_1 is the smallest k in all marked m_{jk}, then set $\underline{p}_{j1}$ to be the smaller one of the two elements in $\overline{m}_{j1k1}$ (or set $\underline{p}_{j1}$ to be $\bar{p}_{j1}$)

Step 6: Delete the j_1^{th} row and k_1^{th} column of M and then delete all the columns that contain marked m_{j1k}, where $k \neq k_1$.

Step 7: In all remained and marked $\overline{m}_{jk}$ find the smallest k and set it to be k_2, then let $\underline{p}_{j2}$ be the smaller one of the two elements in $\overline{m}_{j2k2}$ (or let $\underline{p}_{j2}$ be $\bar{p}_{j2}$).

Step 8: Delete the j_2^{th} row and the k_2^{th} column of M and then delete all the columns that contain marked $\overline{m}_{j2k}$, where $k \neq k_2$.

Step 9: Repeat Step *7 & 8* until no marked m_{jk} remained.

Step10: The other $\underline{p}_j$ which are not set in steps 5 to *8*, are set to be zero.

Lemma 4.3.2

If the interval valued fuzzy relational equation is of the form as (4.3.1), for giving $m \times n$ interval matrix Q and $1 \times n$ interval vector r, the interval minimum solution $\underline{p}$ can be obtained by the above algorithm.

Proof

Owing the fact that the whole interval minimum solution of $\underline{p}$ are derives step by step from the above algorithm. Hence we can prove it in a straight way. The steps *1* and *2* are the standard procedure that has illustrated in section 4.1.1. Since $0 \le \underline{p}_j \le \overline{p}_{j,}$ step *3* is in fact, to put all the possible interval solution elements together. That is, with step *3*, we would not miss any possible interval solution in the solving procedure. Thus, for deriving the whole minimum solutions $\underline{p}_{,}$ the procedure in step *3* is necessary. In step *4,* according to step *3* and Lemma (1.2.15), we have the following deductions. That is after checking $min\ (p_{jL},\ q_{jkL}) = r_{kL}$ and $min\ (p_{jU},\ q_{jkU}) = r_{kU},$ it as noted that the position of m_{jkL} and m_{jkU} make $min\ (\overline{p}_{jL},\ q_{jkL}) = r_{kL}$ $min\ (\overline{p}_{jU},\ q_{jkU}) = r_{KU}$ happen and the results in $\overline{m}_{jkL}$ and $\overline{m}_{jkU}$ must be an element of the minimum solutions. Therefore by Lemma (2.1.2(ii)), $min\ (\overline{p}_j,\ q_{jk}) = r_k$ it as noted that the position of $\overline{m}_{jk}$ make $min\ (\overline{p}_j,\ q_{jk}) = r_k$ happen and the results in $\overline{m}_{jk}$ must be an element of the minimum solutions. Accordingly, we should mark all these elements. In step *5,* we need to check interval minimum solution from high rank. If it is true, by step *4*, pock the minimum solution for the corresponding $p_{j.}$ In step *6:* obviously, since the interval minimum solution for j_1^{th} row of M (ie, j_1^{th} element of $\underline{p}$) has been gotten, we delete all the other columns that contain marked $\overline{m}_{jk,}$ where $k \neq k_{1.}$ That is, the minimum solution for the j_1^{th} element of $\underline{p}$ would not be repeated. The analogous procedure in steps *7* to *9* is to guarantee running of the algorithm from the left to right and from the upper to the bottom in Interval matrix M and to get the whole interval minimum solutions. In step *10,* if we cannot find the interval minimum solution for p_j the zero must be a solution naturally, the analogous proof can be done also for the *max – product* composition.

Remark 4.3.3

For the interval valued fuzzy relational equations as in which P is $s\times m$ matrix and R is $s\times n$ matrix, the solving problem can be seen as to solve s times of the problem as (4.3.1) repeatedly.

Illustrated example 4.3.4

Since the algorithms are similar no matter on solving *max.min* or *max – product* composition we only present an example with *max.min* composition to illustrate the procedure of the algorithm. Consider an interval valued fuzzy relational equation $P{\cdot}Q = R$ with *max.min* composition, where

$$Q = \begin{pmatrix} [0.5,\ 0.7] & [0.7,\ 0.9] & [0,\ 0] & [0,\ 0] \\ [0.4,\ 0.6] & [0.6,\ 0.8] & [0.4,\ 0.6] & [0,\ 0] \\ [0.2,\ 0.4] & [0.4,\ 0.6] & [0.5,\ 0.7] & [0.6,\ 0.8] \\ [0.1,\ 0.3] & [0.2,\ 0.4] & [0,\ 0] & [0.8,\ 1] \end{pmatrix}$$

$$R = \begin{pmatrix} [0.5,\ 0.7] & [0.5,\ 0.7] & [0,\ 0] & [0,\ 0] \end{pmatrix}$$

By our representation (4.1.1) we have ,

$$Q_L = \begin{pmatrix} 0.5 & 0.7 & 0 & 0 \\ 0.4 & 0.6 & 0.4 & 0 \\ 0.2 & 0.4 & 0.5 & 0.6 \\ 0.1 & 0.2 & 0 & 0.8 \end{pmatrix},$$

$$Q_U = \begin{pmatrix} 0.7 & 0.9 & 0 & 0 \\ 0.6 & 0.8 & 0.6 & 0 \\ 0.4 & 0.6 & 0.7 & 0.8 \\ 0.3 & 0.4 & 0 & 1 \end{pmatrix}$$

$$R_L = [\,0.5 \quad 0.5 \quad 0.4 \quad 0\,] \text{ and } R_U = [0.7 \quad 0.7 \quad 0 \quad 0]$$

First our task is to find the minimal solution $\underline{p} \in r^{1x4}$ in equation 4.3.2.

Step1 : It is known that the solution P may be exists (By Section 4.1.1)

Step2 : From section 4.1.1, the maximum solution $p = [\,0.5 \quad 0.5 \quad 0 \quad 0\,]$

Step3 : Build M_L as

$$M_L = \begin{pmatrix} (0.5, 0.5) & (0.5, 0.7) & (0.5, 0) & (0.5, 0) \\ (0.5, 0.4) & (0.5, 0.6) & (0.5, 0.4) & (0.5, 0) \\ (0, 0.2) & (0, 0.4) & (0, 0.5) & (0, 0.6) \\ (0, 0.1) & (0, 0.2) & (0, 0) & (0, 0.8) \end{pmatrix}$$

Step4 : Underline those elements which satisfies $min\,(\overline{p}_j, q_{jk}) = r_k$

$$M_L = \begin{pmatrix} \underline{(0.5, 0.5)} & \underline{(0.5, 0.7)} & (0.5,0) & \underline{(0.5, 0)} \\ (0.5, 0.4) & \underline{(0.5, 0.6)} & \underline{(0.5, 0.4)} & \underline{(0.5,0)} \\ (0, 0.2) & (0, 0.4) & (0, 0.5) & \underline{(0, 0.6)} \\ (0, 0.1) & (0, 0.2) & (0,0.) & \underline{(0, 0.8)} \end{pmatrix}$$

Step5: set $\underline{p}_1 = min\,(0.5, 0.5) = 0.5;$ note here $j = l$

Step6: Delete the first row and the first column of M_L, and then delete all the columns that contain marked m_{lkL} , where $k \neq l$.

$$M_L = \begin{pmatrix} \cancel{(0.5, 0.5)} & \cancel{\underline{(0.5, 0.7)}} & \cancel{(0.5, 0)} & \cancel{\underline{(0.5, 0)}} \\ \cancel{(0.5, 0.4)} & \cancel{\underline{(0.5, 0.6)}} & \underline{(0.5, 0.4)} & \cancel{\underline{(0.5, 0)}} \\ \cancel{(0, 0.2)} & \cancel{(0, 0.4)} & (0, 0.5) & \cancel{\underline{(0, 0.6)}} \\ \cancel{(0, 0.1)} & \cancel{(0, 0.2)} & (0, 0.) & \cancel{\underline{(0, 0.8)}} \end{pmatrix}$$

Step7: Set $\underline{p}_2 = \min(0.5, 0.4) = 0.4;\ here\ j=2$

Step8: Delete the second row and the third column of M_L, and then delete all the columns that contain marked m_{2kL} , where $k \neq 3$.

$$\mathrm{M_L} = \begin{pmatrix} \cancel{(0.5, 0.5)} & \cancel{\underline{(0.5, 0.7)}} & \cancel{(0.5, 0)} & \cancel{\underline{(0.5, 0)}} \\ \cancel{(0.5, 0.4)} & \cancel{\underline{(0.5, 0.6)}} & \cancel{\underline{(0.5, 0.4)}} & \cancel{\underline{(0.5, 0)}} \\ \cancel{(0, 0.2)} & \cancel{(0, 0.4)} & \cancel{(0, 0.5)} & \cancel{\underline{(0, 0.6)}} \\ \cancel{(0, 0.1)} & \cancel{(0, 0.2)} & \cancel{(0, 0.)} & \cancel{\underline{(0, 0.8)}} \end{pmatrix}$$

Step9: Until now, we have set $\underline{p}_1 = 0.5$ and $\underline{p}_2 = 0.4,$ then the other $\underline{p}_j$ are set to be zero, that is $\underline{p}_3 = 0$ and $\underline{p}_4 = 0$. Therefore, we have only one minimal solution of equation (4.1.1) as $\underline{p} = [0.5 \quad 0.4 \quad 0 \quad 0]$.

Similarly, we have to find only one minimal solution of equation (4.1.3) as $\underline{p} = [\,0.7 \quad 0.6 \quad 0 \quad 0\,]$

Therefore, by Lemma (2.1.2(ii)) we have minimal solution of the interval valued fuzzy relational equation $P \cdot Q = R$ as $\underline{p} = \left([0.5, 0.7] \quad [0.4, 0.6] \quad [0, 0] \quad [0, 0]\,.\right)$

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5. APPLICATIONS OF INTERVAL VALUED FUZZY MATRICES

In this chapter, we determine the index and period on an *IVFM* in terms of that of its lower and upper limit fuzzy matrices, which leads to the definition of transitive closure of the concept *IVFM*. We discuss knowledge-based interval valued fuzzy information retrieval method based on concept interval networks by using the transitive closure of the *IVFM* and illustrate with suitable examples. We extend Sanchez's approach for medical diagnosis using the representation of an interval valued fuzzy matrix as an interval matrix of two fuzzy matrices. We propose a method based on Sanchez's approach to study medical diagnosis. In our method the matrix operations involved are *max.min*, which is uniform and simpler than that are found in [8, 10, 40] where intuitionistic fuzzy matrix operations (ie, *max.min, min.max*) are involved in the computation of medical knowledge. We have introduced the arithmetic mean (*am*) matrix of an *IVFM* A as the average of the lower and upper limit matrices A_L and A_U and directly apply Sanchez's method of medical diagnosis for the *am(A),* which is a fuzzy matrix. We propose a new approach to determine the shortest path in Interval Valued Fuzzy Networks (*IVFN)* in which the edges representing the roads connecting the cities and each edge (i, $i+1$) has an associated weight representing the traffic on the road connecting the cities i and $i+1$, which is an interval fuzzy number of the form $R_i = [R_{iL}, R_{iU}]$ for each i and we apply the technique used in [2, 21] to determine the shortest path in lower and upper limits of the fuzzy networks. We have defined the shortest path for an *IVFN* as that path for which the shortest path in lower limit fuzzy network coincides with the shortest path in upper limit fuzzy network and weight = $[w_L, w_U]$ where the w_L and w_U are the weights of the shortest path for lower and upper limit fuzzy networks respectively.

5.1 Document Retrieval Systems

In this section, we compute the index and period for an *IVFM* and find the relations between the index and period of an *IVFM* A with the indices and periods of its lower and upper limit fuzzy matrices A_L and A_U. This leads to the existence of transitive closure of an *IVFM*. We discuss knowledge-based interval valued fuzzy information retrieval method based on concept interval networks by using the transitive closure of the *IVFM* and Illustrate with suitable examples.

Definition 5.1.1

For $A \in (IVFM)_n$, $A^{k+d} = A^k$ holds for some $k, d > 0$, then the least $k > 0$ such that $A^{k+d} = A^k$ for some k is called the index of A, the least $d > 0$ such that $A^{k+d} = A^k$ for some d is called the period of A, denoted as $i(A)$ and $p(A)$ respectively.

Theorem 5.1.2

For $A \in (IVFM)_n$, if $A = [A_L, A_U]$ then $i(A) = max\{i(A_L), i(A_U)\}$ and $p(A) = lcm\ \{p(A_L), p(A_U)\}$.

Proof

Let $i(A) = k$ and $p(A) = d$, then $A^{k+d} = A^k$.

Since $A = [A_L, A_U]$, by Lemma (2.1.2(ii)), $[A_L^{k+d}, A_U^{k+d}] = [A_L^k, A_U^k]$.

Comparing the corresponding blocks we get, $A_L^{k+d} = A_L^k$ and $A_U^{k+d} = A_U^k$.

If $i(A_L) = k_1$, $i(A_U) = k_2$, $p(A_L) = d_1$ and $p(A_U) = d_2$, then by Lemma (1.2.16) we get $k \geq k_1$, $k \geq k_2$ and both $p(A_L) = d_1$ and $p(A_U) = d_2$ divides d.

Therefore $i(A) = k \geq max\ \{k_1, k_2\} = \tilde{\boldsymbol{k}}$ and $lcm\ \{d_1, d_2\}/d = \tilde{\boldsymbol{d}}$ → (5.1.1)

On the other hand, if $\tilde{\boldsymbol{k}}$ is $max\ \{k_1, k_2\}$, then $\tilde{\boldsymbol{k}} \geq k_1$ and k_2 and $\tilde{\boldsymbol{d}}$ is $lcm\ \{d_1, d_2\}/d$

By Lemma (1.2.16), $A_L{}^{\tilde{k}+\tilde{d}} = A_L{}^{\tilde{k}}$ and $A_U{}^{\tilde{k}+\tilde{d}} = A_U{}^{\tilde{k}}$.

By Lemma (2.1.2(ii)) we get, $A^{\tilde{k}+\tilde{d}} = A^{\tilde{k}}$

Again by Lemma (1.2.16), it follows that, $\tilde{\boldsymbol{k}} \geq i(A)$ and $p(A)/\tilde{\boldsymbol{d}}$. → (5.1.2)

From (5.1.1) and (5.1.2) it follows that $i(A) = max\{i(A_L), i(A_U)\}$ and $p(A) = lcm\ \{p(A_L), p(A_U)\}$.

Hence the Theorem.

Now, the Definition (5.1.1) and Theorem (5.1.2) leads to the following definition.

Definition 5.1.3

Let M be an interval valued fuzzy matrix of order n. Then there exist an integer $p \leq n-1$, such that under the composition of interval valued fuzzy matrices $M^p = M^{p+1} = M^{p+2}$ and $T = M^p$ is called the transitive closure of an *IVFM M*. Thus the transitive closure of an *IVFM* $M = [M_L, M_U]$ is the interval matrix whose lower and upper limit fuzzy matrices are the transitive closure of M_L and M_U, that is, $M = [M_L, M_U]$ then $T = [T_L, T_U]$ is the transitive closure of M.

$$T = M^p$$

$$[T_L, T_U] = [M_L{}^p, M_U{}^p] \quad \text{(By Lemma (2.1.2(ii)))}$$

$$T_L = M_L{}^p \text{ and } T_U = M_U{}^p.$$

Interval Valued Concept Networks

A concept interval network includes nodes and directed links. Each node represents a concept (or) a document. Each directed link connects concept to concept (or) directs from one concept to a document. Let us consider an interval network with n concepts $\{ c_1, c_2, c_3 \ldots\ldots c_n\}$ and m documents $\{d_1, d_2, \ldots\ldots d_m\}$.

If $c_i \xrightarrow{\mu} c_j$, then it indicates that the degree of relevance from concept c_i to concept c_j is μ, where μ is a subinterval of $[0, 1]$. If $c_i \xrightarrow{\mu} d_j$, then it indicates that the degree of relevance of document d_j with respect to the concept c_i is μ, where $\mu = c_{ij} =[c_{ijL}, c_{ijU}]$ → (5.1.3)

is an interval of $[0, 1]$. The relevant interval value from concept c_i to concept c_j and the relevant interval value from concept c_j to concept c_k are given, that is, c_{ij} and c_{ik} are known and c_{ik} is defined as follows:

$$c_{ik} = min\ \{c_{ij}, c_{jk}\} \qquad \rightarrow (5.1.4)$$

$$[c_{ikL}, c_{ikU}] = min\ \{[c_{ijL}, c_{ijU}], [c_{jkL}, c_{jkU}]\} \qquad \text{(By (5.1.3))}$$

$$c_{ikL} = min\ \{c_{ijL}, c_{jkL}\}\ and\ c_{ikU} = min\ \{c_{ijU}, c_{jkU}\} \quad \text{(By Lemma (2.1.2(ii)))} \rightarrow (5.1.5)$$

Similarly, if $c_{12}, c_{23} \ldots\ldots c_{(n-1)n}$ are known, then, → (5.1.6)

By (5.1.3) and Lemma (2.1.2(ii)) we have, $c_{1nL} = min\ \{c_{12L}, c_{23L} \ldots\ldots c_{(n-1)nL}\}$ and

$$c_{1nU} = min\ \{c_{12U}, c_{23U} \ldots\ldots c_{(n-1)nU}\} \qquad \rightarrow (5.1.7)$$

Definition 5.1.4

Let $\{c_1, c_2, \ldots, c_n\}$ be a set of n concepts. A concept interval valued fuzzy matrix $C = (c_{ij})$ is an $n \times n$ interval valued fuzzy matrix, where c_{ij} is the relevant interval value from the concept c_i to the concept c_j and c_{ij} is a subinterval of $[0, 1]$ satisfying the following properties.

(i) Reflexivity: $c_{ii} = [1, 1]$ for each $i = 1$ to n

$$[c_{iiL}, c_{iiU}] = [1, 1] \qquad \text{(By (5.1.3))}$$

$$c_{iiL} = 1 \text{ and } c_{iiU} = 1$$

for each $i = 1$ to n.

(ii) Non-Symmetric: $c_{ij} \neq c_{ji}$

$$[c_{ijL}, c_{ijU}] \neq [c_{jiL}, c_{jiU}] \qquad \text{(By (5.1.3))}$$

$$c_{ijL} \neq c_{jiL} \text{ and } c_{ijU} \neq c_{jiU}.$$

(iii) Transitivity: $c_{ik} \geq max_j\, min\{c_{ij}, c_{jk}\}$

$$[c_{ikL}, c_{ikU}] \geq max_j\, min\{[c_{ijL}, c_{ijU}]. [c_{jkL}, c_{jkU}]\} \qquad \text{(By (5.1.3))}$$

$$c_{ikL} \geq max_j\, min\, \{c_{ijL}, c_{jkL}\} \text{ and } c_{ikU} \geq max_j\, min\, \{c_{ijU}, c_{jkU}\}$$

Definition 5.1.5

Let $\{d_1, d_2, \ldots d_m\}$ be a set of documents and $\{c_1, c_2, \ldots, c_n\}$ be a set of concepts in a concept interval network with m documents and n concepts. A document descriptor interval matrix $D = (d_{ij})$ is an $m \times n$ matrix, where d_{ij} is the degree of relevance of document d_i with respect to the concept c_j.

The document descriptor interval valued fuzzy matrix $D^* = DT$, where D is the document descriptor of the interval network and T is the transitive closure of the concept interval matrix. By Lemma (2.1.2(ii)), $D_L{}^* = D_{L.}T_L$ and $D_U{}^* = D_UT_{U\cdot}$ Indicates the degree of relevance of each document with respect to specific concepts and is used as a basis for similarity measures between queries and documents. We shall illustrate the above basic concepts in a concept interval network with suitable examples.

Let us illustrate the concept *IVFM*, Query descriptor, Document descriptor *IVFM* and compute the transitive closure on an *IVFM* in the following examples.

Illustration 5.1.6

We consider a network $N = (V, E)$ consisting n nodes (cities) and m edges (roads) connecting the cities of a country. If we measure the vehicles an the roads of the network for a particular time duration, it is quite impossible to measure the vehicles on a road as a single value because the vehicles in a duration is not fixed, it varies from time to time. So, appropriate technique to gradation of vehicles is an interval and not a point. In this case, the network (5.1.1) is called interval valued fuzzy networks.

Let us consider a concept interval network in Figure 5.1.1 where $c_1, c_2, \ldots c_n$ are concepts, d_1, d_2, d_3 are the documents. If the query descriptor Q is $Q = \{(c_3, [1, 1])\}$ where $[1, 1]$ represents the relevant interval fuzzy value of the query descriptor Q with respect to the concept c_3, then the relevant interval fuzzy value of document d_2 with respect to the concept c_3 is calculated as follows:

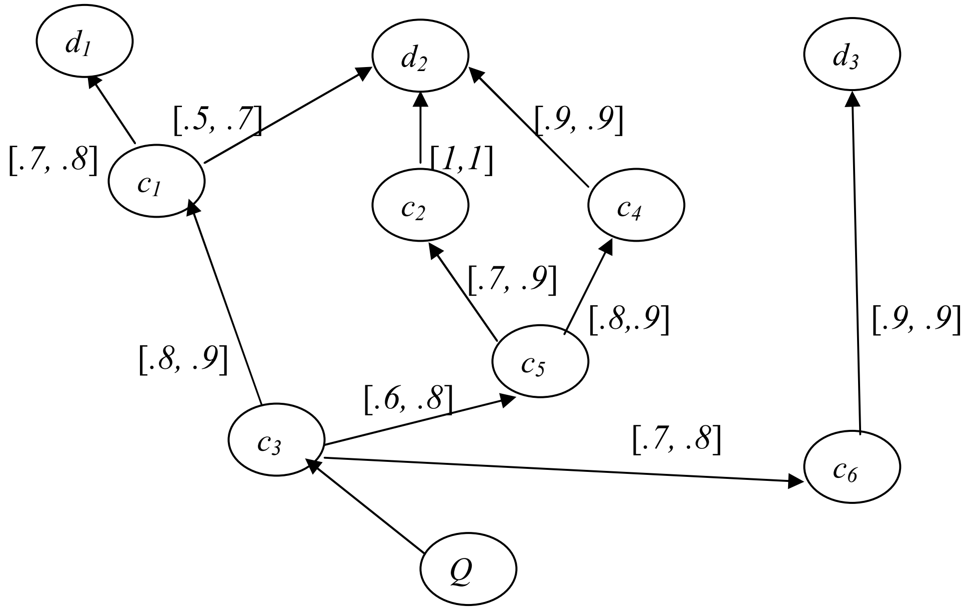


Figure 5.1.1

From Figure 5.1.1 we see there are three distinct routes from the concept c_3 to the document d_2

1. The first route is $c_3 \xrightarrow{[.8, .9]} c_1 \xrightarrow{[.5, .7]} d_2$. The relevant interval fuzzy value of the document 'd_2' with respect to concept c_3 is calculated by using (5.1.4) as $\min \{[.8, .9], [.5, .7]\} = [.5, .7]$.

2. The second route is $c_3 \xrightarrow{[.6, .8]} c_5 \xrightarrow{[.7, .9]} c_2 \xrightarrow{[1, 1]} d_2$. The relevant interval fuzzy value of the document 'd_2' with respect to concept c_3 is $\min \{[.6, .8], [.7, .9], [1, 1]\} = [.6, .8]$.

3. The third route is $c_3 \xrightarrow{[.6, .8]} c_5 \xrightarrow{[.8, .9]} c_4 \xrightarrow{[.9, .9]} d_2$. The relevant interval fuzzy value of the document 'd_2' with respect to concept c_3 is $\min \{[.8, .9], [.8, .9], [.9, .9]\} = [.6, .8]$.

Then the relevant interval value of the document d_2 with respect to the concept c_3 is $max\ \{[.5, .7], [.6, .8], [.6, .8]\} = [.6, .8]$.

Thus $Q = (c_3, [1, 1]) = [.6, .8]$.

Example 5.1.7

The concept interval valued matrix C of the interval network in Figure (5.1.1) is calculated by using (5.1.4), (5.1.5) and (5.1.6).

$$C = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{array} \begin{array}{c} \begin{array}{cccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array} \\ \begin{pmatrix} [1, 1] & [0, 0] & [0, 0] & [0, 0] & [0, 0] & [0, 0] \\ [0, 0] & [1, 1] & [0, 0] & [0, 0] & [0, 0] & [0, 0] \\ [.8, .9] & [.6, .8] & [1, 1] & [.6, .8] & [.6, .8] & [.7, .8] \\ [0, 0] & [0, 0] & [0, 0] & [1, 1] & [0, 0] & [0, 0] \\ [0, 0] & [.7, .9] & [0, 0] & [.8, .9] & [1, 1] & [0, 0] \\ [0, 0] & [0, 0] & [0, 0] & [0, 0] & [0, 0] & [1, 1] \end{pmatrix} \end{array}$$

By our representation (2.1.1) we have, $C = [C_L, C_U]$

$$C_L = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{array} \begin{array}{c} \begin{array}{cccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ .8 & .6 & 1 & .6 & .6 & .7 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & .7 & 0 & .8 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \text{ and}$$

$$C_U = \begin{array}{c|cccccc} & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ \hline c_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ c_2 & 0 & 1 & 0 & 0 & 0 & 0 \\ c_3 & .9 & .8 & 1 & .8 & .8 & .8 \\ c_4 & 0 & 0 & 0 & 1 & 0 & 0 \\ c_5 & 0 & .9 & 0 & .9 & 1 & 0 \\ c_6 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

From the concept fuzzy matrix C_L and C_U we see that all the diagonal entries are 1. That is, $c_{iiL} = 1$ for $i = 1$ to 6. Here $c_{23L} = 0$ but $c_{32L} = 0.6$ and therefore $c_{ijL} \neq c_{jiL}$ and $c_{iiU} = 1$ for $i = 1$ to 6. Here $c_{23U} = 0$ but $c_{32U} = 0.8$ and therefore $c_{ijU} \neq c_{jiU}$

Hence, by Lemma (2.1.2(ii)) we have, $c_{ii} = [1, 1]$ for $i = 1$ to 6. Here $c_{23} = [0, 0]$ but $c_{32} = [.6, .8]$ and therefore $c_{ij} \neq c_{ji}$

Hence C is not symmetric.

Example 5.1.8

The document descriptor interval valued fuzzy matrix D for the concept interval network in Figure (5.1.1) is computed as in Illustration (5.1.6).

$$D = \begin{array}{c|cccccc} & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ \hline d_1 & [.7, .8] & [0, 0] & [.7, .8] & [0, 0] & [0, 0] & [0, 0] \\ d_2 & [.5, .7] & [1, 1] & [.6, .8] & [.9, .9] & [.8, .9] & [0, 0] \\ d_3 & [0, 0] & [0, 0] & [.7, .8] & [0, 0] & [0, 0] & [.9, .9] \end{array}$$

By our representation (2.1.1) we have, $D = [D_L, D_U]$

$$D_L = \begin{array}{c|cccccc} & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ \hline d_1 & .7 & 0 & .7 & 0 & 0 & 0 \\ d_2 & .5 & 1 & .6 & .9 & .8 & 0 \\ d_3 & 0 & 0 & .7 & 0 & 0 & .9 \end{array} \quad \text{and } D_U = \begin{array}{c|cccccc} & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ \hline d_1 & .8 & 0 & .8 & 0 & 0 & 0 \\ d_2 & .7 & 1 & .8 & .9 & .9 & 0 \\ d_3 & 0 & 0 & .8 & 0 & 0 & .9 \end{array}$$

Then *'0'* entries in the D_L and D_U *i*ndicate that the corresponding concepts are not relevant (or) can be neglected with respect to the particular document. For instance the concepts c_2, c_4, c_5 and c_6 are not relevant for the document ' d_1' in D_L and D_U. To get the implicit relevant values of each document with respect to specific concepts, let us compute the transitive closure of the concept matrix given in Example (5.1.7) for the concept interval network in Figure (5.1.1).

$$C_L^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ .8 & .6 & 1 & .6 & .6 & .7 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & .7 & 0 & .8 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ .8 & .6 & 1 & .6 & .6 & .7 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & .7 & 0 & .8 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ .8 & .6 & 1 & .6 & .6 & .7 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & .7 & 0 & .8 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$= C_L$

Similarly, $C_U^2 = C_U$.

By Lemma (2.1.2(ii)) we have, $C = C^2$. Therefore C itself is the transitive closure of the concept interval valued fuzzy matrix. Since $C = T$, $T_L = C_L$ and $T_U = C_U$

$$D_L{*} = D_L\, T_L = \begin{pmatrix} .7 & 0 & .7 & 0 & 0 & 0 \\ .5 & 1 & .6 & .9 & .8 & 0 \\ 0 & 0 & .7 & 0 & 0 & .9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ .8 & .6 & 1 & .6 & .6 & .7 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & .7 & 0 & .8 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} .7 & .6 & .7 & .6 & .6 & .7 \\ .6 & 1 & .6 & .9 & .8 & .6 \\ .7 & .6 & .7 & .6 & .6 & .9 \end{pmatrix} \text{ and}$$

$$D_U{*} = D_U.T_U = \begin{pmatrix} .8 & .8 & .8 & .8 & .8 & .8 \\ .8 & 1 & .8 & .9 & .9 & .8 \\ .8 & .8 & .8 & .8 & .8 & .9 \end{pmatrix}$$

By Lemma (2.1.2(ii)) we have,

$$D{*} = D\,T = \begin{pmatrix} [.7, .8] & [.6, .8] & [.7, .8] & [.6, .8] & [.6, .8] & [.7, .8] \\ [.6, .8] & [1, 1] & [.6, .8] & [.9, .9] & [.8, .9] & [.6, .8] \\ [.7, .8] & [.6, .8] & [.7, .8] & [.6, .8] & [.6, .8] & [.9, .9] \end{pmatrix}$$

The document descriptor interval valued fuzzy matrix $D*$ gives the implicit values of each document more accurately. For instance '$[0, 0]$' entries in the first row of the document descriptor *IVFM* in Example (5.1.8) are improved as $[.6, .8]$, $[.6, .8]$, $[.6, .8]$ and $[.7, .8]$ respectively that is, the concepts c_2, c_4, c_5 and c_6 cannot be neglected for the document d_1. Thus the document descriptor interval valued fuzzy matrix $D*$ obtained by using the transitive closure '*I* 'reveals more accurate results for the user.

5.2 Medical Diagnosis

In this section, we extend Sanchez's approach for medical diagnosis using the representation of an interval valued fuzzy matrix as an interval matrix of two fuzzy matrices. We introduce arithmetic mean of an interval valued fuzzy matrix as the arithmetic mean of its lower and upper limit matrices and propose a method to study Sanchez's approach of medical diagnosis through the arithmetic mean of an interval valued fuzzy matrix, which is a simpler technique than that of using intuitionistic fuzzy sets available in the literature.

Suppose S is a set of symptoms of certain diseases, D is a set of diseases and P is a set of patients, construct an Interval valued fuzzy matrix (F, D) over S, where F is a mapping $F: D \rightarrow \tilde{F}(S)$, $\tilde{F}(S)$ is a set of all interval valued fuzzy sets of S. A relation matrix say, R_1 is constructed from the interval valued fuzzy matrix (F, D) and called symptom - disease matrix. Similarly its compliment $(F, D)^c$ gives another relation matrix, say R_2, called non symptom diseases matrix. Analogous to Sanchez's notion of medical knowledge, we refer to each of the matrices R_1 and R_2 as medical knowledge of an interval valued fuzzy matrix. Again we construct another interval valued fuzzy matrix (F_1, S) over P, where F_1 is a mapping given by $F_1: S \rightarrow \tilde{F}(P)$. This Interval valued fuzzy matrix gives another relation matrix Q called patient-symptom matrix. Then we obtain two new relation matrices $T_1 = Q\,R_1$ and $T_2 = Q\,R_2$ called symptom patient matrix and non symptom patient matrix respectively.

Now,

$$T_1 = Q\,R_1 \qquad \rightarrow (5.2.1)$$

$$T_2 = Q\,R_2 \qquad \rightarrow (5.2.2)$$

Let $T_1 = [T_{1L}, T_{1U}]$, $Q = [Q_L, Q_U]$, $R_1 = [R_{1L}, R_{1U}]$, $T_2 = [T_{2L}, T_{2U}]$, and $R_2 = [R_{2L}, R_{2U}]$, be the representation of the form (2.1.1) for the *IVFM* T_1, Q, R_1, T_2 and R_2. Then by using the *IVFM* operation By Lemma (2.1.2(ii)) in (5.2.1) and (5.2.2) we get,

$$T_{1L} = Q_L R_{1L} \text{ and } T_{1U} = Q_U R_{1U} \qquad \rightarrow (5.2.3)$$

$$T_{2L} = Q_L R_{2L} \text{ and } T_{2U} = Q_U R_{2U} \qquad \rightarrow (5.2.4)$$

Let us define the non-disease matrices T_{3L}, T_{3U}, T_{4L} and T_{4U} Corresponding to T_{1L}, T_{1U} T_{2L}, and T_{2U} respectively as

$$T_{3L} = Q_L \cdot (J - R_{1L}) \text{ and } T_{3U} = Q_U \cdot (J - R_{3U}) \qquad \rightarrow (5.2.5)$$

$$T_{4L} = Q_L \cdot (J - R_{2L}) \text{ and } T_{4U} = Q_U \cdot (J - R_{2U}) \qquad \rightarrow (5.2.6)$$

Where J is the matrix with all entries '*1*'. Now,

$$S_{T1L} = \max_{i,j} [T_{1L}(p_i, d_j), T_{4L}(p_i, d_j)] \text{ and } S_{T1U} = \max_{i,j} [T_{1U}(p_i, d_j), T_{4U}(p_i, d_j)] \quad \forall i = 1,2,3 \text{ and } j = 1,2. \qquad \rightarrow (5.2.7)$$

$$S_{T2L} = \max_{i,j} [T_{2L}(p_i, d_j), T_{3L}(p_i, d_j)] \text{ and } S_{T2U} = \max_{i,j} [T_{2U}(p_i, d_j), T_{3U}(p_i, d_j)] \quad \forall i = 1,2,3 \text{ and } j = 1,2. \qquad \rightarrow (5.2.8)$$

We calculate the diagnosis score S_{T1} and S_{T2} for and against the diseases respectively

$$S_{T1} = \max_{i,j} \left[S_{T1U}(p_i, d_j), S_{T2L}(p_i, d_j) \right] \quad \forall i = 1,2,3 \text{ and } j = 1,2 \qquad \rightarrow (5.2.9)$$

$$S_{T2} = \max_{i,j} \left[S_{T1L}(p_i, d_j), S_{T2U}(p_i, d_j) \right] \quad \forall i = 1,2,3 \text{ and } j = 1,2 \qquad \rightarrow (5.2.10)$$

Now, if $\max_j \left[S_{T1}(p_i, d_j) - S_{T2}(p_i, d_j) \right]$ $\rightarrow$ (5.2.11)

Occurs for exactly $(p_i\ ,\ d_k)$ only, then we conclude that the acceptable diagnostic hypothesis for patient p_i is the disease d_k. In case there is a tie, the process has to be repeated for patient p_i by reassessing the symptoms.

Algorithm 5.2.1

1. Input the interval valued fuzzy matrices (F, D) and $(F, D)^C$ over the set S of symptoms S , where D is the set of diseases. Also write the medical knowledge matrix R_1 and R_2 representing the relation matrices of the $IVFM\ (F, D)$ and $(F, D)^c$ respectively.
2. $R_2 = 1\text{-}R_1 = [1\text{-}R_{IU}, 1\text{-}R_{IL}]$
3. Input the $IVFM\ (F_1, S)$ over the set P of patients and write its relation matrix Q.
4. Compute the relation matrices
   (i) $T_{1L} = Q{\cdot}R_{IL}$ and $T_{1U} = Q_U{\cdot}R_{IU}$
   (ii) $T_{2L} = Q_L{\cdot}R_{2L}$ and $T_{2U} = Q_U{\cdot}R_{2U}$ then we get,
   $T_{3L} = Q_L{\cdot}\ (J– R_{1L})$ and $T_{3L} = Q_U{\cdot}(J– R_{1U})$
   $T_{4L} = Q_L{\cdot}(J– R_{2L})$ and $T_{4L} = Q_U{\cdot}(J– R_{2U})$
5. Compute $S_{T1L,}\ S_{T1U}$, S_{T2L} and $S_{T2U.}$
6. Compute the diagnosis scores ST_1 and $ST_{2.}$
7. Find $S_K = max_j\ [S_{T1}(p_i, d_j) – S_{T2}(p_i, d_j)]$ then we conclude that the patient p_i is suffering from the disease d_k.
8. If S_k has more than one value then go to step one and repeat the process by reassessing the symptoms for the patient

Now we apply our technique for the same hypothetical case study presented in [8].

Illustration 5.2.2

Suppose there are three patients' p_1, p_2 and p_3 in a hospital with symptoms temperature, headache, cough and stomach problem. Let the possible diseases relating to the above symptoms be viral fever and malaria. We consider the set $S = \{e_1, e_2, e_3, e_4\}$ as universal set, where e_1, e_2, e_3 and e_4 represent the symptoms temperature, headache, cough and stomach problem respectively and the set $D = \{d_1, d_2\}$ where d_1 and d_2 represent the parameters viral fever and malaria respectively. Suppose that $F(d_1) = [<e_1, [.7, 1]>, <e_2, [.1, .4]>, <e_3, [.5, .6]>, <e_4, [.2, .4]>]$, $F(d_2) = [<e_1, [.6, .9]>, <e_2, [.4, .6]>, <e_3, [.3, .6]>, <e_4, [.8, 1]>]$. The Interval valued fuzzy matrix (F, D) is a parameterized family $[F(d_1), F(d_2)]$ of all interval valued fuzzy matrix over the set S and are determined from expert medical documentation. Thus the fuzzy matrix (F, D) gives an approximate description of the interval valued fuzzy matrix medical knowledge of the two diseases and their symptoms. This interval valued fuzzy matrix (F, D) and its complement $(F, D)^C$ are represented by two relation matrices R_1 and R_2 called symptom – disease matrix and non symptom disease matrix respectively given by

$$R_1 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} [.7, 1] & [.6, .9] \\ [.1, .4] & [.4, .6] \\ [.5, .6] & [.3, .6] \\ [.2, .4] & [.8, .1] \end{pmatrix} \end{array} \quad \text{and} \quad R_2 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} [.0, .3] & [.1, .4] \\ [.6, .9] & [.4, .6] \\ [.4, .5] & [.4, .7] \\ [.6, .8] & [.0, .2] \end{pmatrix} \end{array}$$

By our representation (2.1.1) we have,

$$R_{1L} = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .7 & .6 \\ .1 & .4 \\ .5 & .3 \\ .2 & .8 \end{pmatrix} \end{array} \text{ and } \quad R_{1U} = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 1 & .9 \\ .4 & .6 \\ .6 & .6 \\ .4 & 1 \end{pmatrix} \end{array}$$

$$R_{2L} = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .0 & .1 \\ .6 & .4 \\ .4 & .4 \\ .6 & .0 \end{pmatrix} \end{array}, \quad R_{2U} = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .3 & .4 \\ .9 & .6 \\ .5 & .7 \\ .8 & .2 \end{pmatrix} \end{array}$$

Again we take $P = \{p_1, p_2, p_3\}$ as the universal set where p_1, p_2 and p_3 represent patients respectively and $S = \{e_1, e_2, e_3, e_4\}$ as the set of parameters suppose that,

$F_1(e_1) = [<p_1, [.6, .9]>, <p_2, [.3, .5]>, <p_3, [.6, .8]>]$,

$F_2(e_2) = [<p_1, [.3, .5]>, <p_2, [.3, .7]>, <p_3, [.2, .6]>]$,

$F_3(e_3) = [<p_1, [.8, 1]>, <p_2, [.2, .4]>, <p_3, [.5, .7]>]$ and

$F_4(e_4) = [<p_1, [.6, .9]>, <p_2, [.3, .5], <p_3, [.2, .5]>]$

The Interval valued fuzzy matrix (F_1, S) is another parameterized family of all interval valued fuzzy matrices and gives a collections of approximate description of the patient-symptoms in the hospital. This interval valued fuzzy matrix (F_1, S) represents a relation matrix Q called patient-symptom matrix given by

$$Q = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} [.6, .9] & [.3, .5] & [.8, 1] & [.6, .9] \\ [.3, .5] & [.3, .7] & [.2, .4] & [.3, .5] \\ [.6, .8] & [.2, .6] & [.5, .7] & [.2, .5] \end{pmatrix} \end{array}$$

By our representation (2.1.1) we have, $Q = [Q_L\,, Q_U]$

$$Q_L = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} .6 & .3 & .8 & .6 \\ .3 & .3 & .2 & .3 \\ .6 & .2 & .5 & .2 \end{pmatrix} \end{array}$$

and

$$Q_U = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} .9 & .5 & 1 & .9 \\ .5 & .7 & .4 & .5 \\ .8 & .6 & .7 & .5 \end{pmatrix} \end{array}$$

Then combining the relation matrices R_{1L}, R_{1U} and R_{2L}, R_{2U} separately with Q_L and Q_U we get the matrices $T_1 = [T_{1L,} T_{1U}]$ and $T_2 = [T_{2L,} T_{2U}]$.

From equations (5.2.3), (5.2.4), (5.2.5) and (5.2.6) we have,

$$T_{1L} = Q_L \cdot R_{1L} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .6 & .6 \\ .3 & .3 \\ .6 & .6 \end{pmatrix} \end{array} \quad \text{and} \quad T_{1U} = Q_U \cdot R_{1U} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .9 & .9 \\ .5 & .6 \\ .8 & .8 \end{pmatrix} \end{array}$$

$$T_{2L} = Q_L \cdot R_{2L} = \begin{array}{c} \\ p_1 \\ p_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .6 & .4 \\ .3 & .3 \\ .4 & .4 \end{pmatrix} \end{array} \quad \text{and} \quad T_{2U} = Q_U \cdot R_{2U} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .8 & .7 \\ .7 & .6 \\ .6 & .7 \end{pmatrix} \end{array}$$

$$T_{3L} = Q_L \cdot (J - R_{1L}) = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .6 & .7 \\ .3 & .3 \\ .5 & .5 \end{pmatrix} \end{array} \text{and} \quad T_{3U} = Q_U \cdot (J - R_{1U}) = \begin{array}{c} \\ p_1 \\ p_2 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .6 & .4 \\ .6 & .4 \\ .6 & .4 \end{pmatrix} \end{array}$$

$$T_{4L} = Q_L\cdot(J-R_{2L}) = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .6 & .6 \\ .3 & .3 \\ .6 & .6 \end{pmatrix} \end{array} \text{ and } T_{4U} = Q_U\cdot(J-R_{2U}) = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .7 & .8 \\ .5 & .5 \\ .7 & .6 \end{pmatrix} \end{array}$$

Now, from equations (5.2.7) and (5.2.8) we have,

$$S_{TIL} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .6 & .6 \\ .3 & .3 \\ .6 & .6 \end{pmatrix} \end{array} \text{ and } S_{T1U} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .9 & .9 \\ .5 & .6 \\ .8 & .8 \end{pmatrix} \end{array}$$

$$S_{T2L} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .6 & .7 \\ .3 & .3 \\ .5 & .5 \end{pmatrix} \end{array} \text{ and } S_{T2U} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .8 & .7 \\ .7 & .6 \\ .6 & .7 \end{pmatrix} \end{array}$$

We calculate the diagnosis score for and against the diseases S_{T1} and S_{T2} from equations (5.2.9) and (5.2.10) we have,

$$S_{T1} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .9 & .9 \\ .5 & .6 \\ .8 & .8 \end{pmatrix} \end{array} \text{ and } S_{T2} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} .8 & .7 \\ .7 & .6 \\ .6 & .7 \end{pmatrix} \end{array}$$

Now, for equation (5.2.1) we have the difference for and against the diseases by *IVFM* Method

$S_{T1} - S_{T2}$	d_1	d_2
p_1	.1	.2
p_2	–.2	.0
p_3	.2	.1

We conclude the patient p_3 is suffering from the disease d_1 and patient's p_1 and p_2 both suffering from disease d_2.

The same conclusion is available in [8], in which matrix computations involved are not uniform.

Arithmetic mean method

In this method, we apply Sanchez's method of medical diagnosis for the arithmetic mean of interval valued fuzzy matrix.

Definition 5.2.3

Let $A = [A_L, A_U]$.

Arithmetic mean of an *IVFM* A = arithmetic mean of A_L and A_U denoted as $am(A)$ is Defined *as* $am(A) = \frac{A_L + A_U}{2} = \left(\frac{a_{ijL} + a_{ijU}}{2}\right)$ is the fuzzy matrix.

The relation matrices R_1, R_2 and Q are constructed as in step *1, 2, 3* of the algorithm (5.2.1). By using the Definition (5.2.3) of the arithmetic mean of an *IVFM,* let us compute the $am(R_1)$, $am(R_2)$ and $am(Q)$ for the matrices $R_1 = [R_{1L}, R_{2U}]$, $R_2 = [R_{2L}, R_{2U}]$ and $Q = [Q_L, Q_U]$. By using Definition (5.2.3),

$$am(R_1) = \frac{R_{1L} + R_{1U}}{2}, \qquad \rightarrow (5.2.12)$$

$$am(R_2) = \frac{R_{2L} + R_{2U}}{2} \quad \rightarrow (5.2.13)$$

and $$am\,(Q) = \frac{Q_L + Q_U}{2} \quad \rightarrow (5.2.14)$$

Then combining the relation matrices $am(R_1)$ and $am(R_2)$ separately with $am(Q)$ under the *max.min* composition of fuzzy matrices we get

$$T_1 = am(Q)\cdot am(R_1) \quad \rightarrow (5.2.15)$$

$$T_2 = am(Q)\cdot am(R_2) \quad \rightarrow (5.2.16)$$

$$T_3 = am(Q)\cdot (J - am(R_1)) \quad \rightarrow (5.2.17)$$

$$T_4 = am(Q)\cdot (J - am(R_2)) \quad \rightarrow (5.2.18)$$

where J is the matrix with all entries '*1*'.

By using Sanchez's technique [38, 39], we calculate the diagnosis score S_{T1} and S_{T2} for and against the disease respectively.

$$S_{T1} = \max_{i,j} [\, T_1 (p_i, d_j) , T_4 (p_i, d_j) \,] \quad \forall i = 1,2,3 \text{ and } j = 1,2. \quad \rightarrow (5.2.19)$$

$$S_{T2} = \max_{i,j} [\, T_2 (p_i, d_j) , T_3 (p_i, d_j) \,] \quad \forall i = 1,2,3 \text{ and } j = 1,2 \quad \rightarrow (5.2.20)$$

Now, if $\max_j [\, S_{T1} (p_i, d_j) - S_{T2} (p_i, d_j) \,]$ →(5.2.21)

Occurs for exactly (p_i, d_i) only, then we conclude that the acceptable diagnostic hypothesis for patient p_i is the disease d_k. In case there is a tie, the process has to be repeated for patient p_i by reassessing the symptoms. Let us illustrate the am method by considering the case study in Illustration 5.2.2.

Example 5.2.4

We shall calculate the Average symptom disease matrix $am(R_1)$, average non symptom disease matrix $am\,(R_2)$ and average patient symptom matrix $am(Q)$ by using (5.2.12), (5.2.13) and (5.2.14) for the matrices R_1, R_2 and Q respectively.

$$am(R_1) = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 0.85 & 0.75 \\ 0.25 & 0.5 \\ 0.55 & 0.45 \\ 0.3 & 0.45 \end{pmatrix} \end{array} \quad \rightarrow (5.2.22)$$

$$am(R_2) = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 0.15 & 0.25 \\ 0.75 & 0.5 \\ 0.45 & 0.55 \\ 0.7 & 0.1 \end{pmatrix} \end{array} \quad \rightarrow (5.2.23)$$

and

$$am(Q) = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} 0.75 & 0.4 & 0.9 & 0.75 \\ 0.4 & 0.5 & 0.3 & 0.4 \\ 0.7 & 0.4 & 0.6 & 0.3 \end{pmatrix} \end{array} \quad \rightarrow (5.2.24)$$

Then combining the relation matrices *am*(R_1) and *am*(R_2) separately with *am*(*Q*) we have

$$T_1 = (amQ) . (am\,R_1) = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 0.75 & 0.75 \\ 0.4 & 0.5 \\ 0.7 & 0.7 \end{pmatrix} \end{array} \quad \rightarrow (5.2.25)$$

$$T_2 = (amQ) . (am\,R_2) = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 0.7 & 0.55 \\ 0.5 & 0.5 \\ 0.7 & 0.55 \end{pmatrix} \end{array} \quad \rightarrow (5.2.26)$$

$$T_3 = (amQ)\,.(J - amR_1) = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 0.7 & 0.55 \\ 0.5 & 0.5 \\ 0.45 & 0.55 \end{pmatrix} \end{array} \qquad \rightarrow (5.2.27)$$

$$T_4 = (amQ)\,.(J - am\,R_2) = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 0.75 & 0.75 \\ 0.4 & 0.5 \\ 0.7 & 0.7 \end{pmatrix} \end{array} \qquad \rightarrow (5.2.28)$$

Then by equations (5.2.19) and (5.2.20) we have,

$$S_{T1} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 0.75 & 0.75 \\ 0.4 & 0.5 \\ 0.7 & 0.7 \end{pmatrix} \end{array} \quad \text{and} \quad S_{T2} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 0.7 & 0.55 \\ 0.5 & 0.5 \\ 0.45 & 0.55 \end{pmatrix} \end{array}$$

Now we calculate from equation (5.2.21) , the difference for and against the diseases by am Method

$S_{T1} - S_{T2}$	d_1	d_2
p_1	0.05	0.15
p_2	-0.1	0
p_3	0.25	0.15

Now, we conclude that the patient p_3 is suffering from the disease d_1 and patient p_1 and p_2 both suffering from the disease d_2.

5.3 Shortest Path Networks

In this section, we propose a new approach to determine the shortest path in an Interval valued fuzzy networks *(IVFN),* a network in which vertices (or nodes) and edges (or links) remain crisp but each edge $(i, i+1)$ has an associated weight, which is an interval fuzzy number of the form $R_i = [R_{iL}, R_{iU}]$ for each i. For each *IVFN*, we associate two fuzzy networks called lower and upper limit fuzzy networks having the same set of vertices and edges but each edge$(i, i+1)$ is attached with a fuzzy weight R_{iL} *a*nd R_{iU} respectively. We exhibit that the shortest path of weight $w = [w_L, w_U]$ an interval fuzzy number in *IVFN,* is that path for which the shortest path of weight w_L in the lower limit fuzzy network coincides with the shortest path of weight w_U in the upper limit fuzzy network. The concept is illustrated with the help of a simple situation and the validation of mathematical verification is provided.

An interval fuzzy network includes nodes and directed links. Each node represents a city. Each directed links $(i, i+1)$ connects city i to $i+1$. Let $X_i = \{X_1, X_2 ,\ldots.. X_{i-1}\}$ denotes the vector of decision variable at stage i and $S_i = \{S_1, S_2 ,\ldots.. S_{i-1}\}$ is the input to stage i-1, f_{i-1} denotes the fuzzy optimal value of the objective function corresponding to the last i-1 stages.

If $X_i \xrightarrow{R_i} S_{i+1}$, then it indicates that the degree of relevance from stage i to stage $i+1$ is R_i where R_i is a sub interval of $[0,1]$. Let $R_i = [R_{iL}, R_{iU}]$ $\rightarrow(5.3.1)$

Since R_i is an interval of [0, 1], $R_{iL} \leq R_{iU.}$ $R_i\, (X_i, S_{i+1})$ is the weight of the corresponding arc $(i, i+1)$. For this interval valued network *(IVFN),* let us construct two networks which we call as lower limit fuzzy network $(FN)_L$ and upper limit fuzzy network $(FN)_U$ with the same set of nodes and links, the weight of the corresponding arc $(i, i+1)$ in the lower limit fuzzy network is R_{iL} and in the upper limit fuzzy network in R_{iU}.

Dynamic programming (*DP*) formulation for the shortest path problem is given as in [2]:

$$f_i(S_{i+1}) = \min_{X_i} (R_i (X_i, S_{i+1}) + f_{i-1}(S_i)) \quad \rightarrow (5.3.2)$$

Through Belman's principle of optimality this recursion (5.3.2) is very flexible and has many applications. One obvious flexibility is that the sum in (5.3.2) can be replaced by almost any binary operator and the recursion will hold in [21]. For the fuzzy optimization problems under the *max.min* composition, the sum in (5.3.2) is the fuzzy addition and (5.3.2) is reformulated as

$$f_i(S_{i+1}) = \min_{X_i} \{max[R_i(X_i, S_{i+1}), f_{i-1}(S_i)]\} \quad \rightarrow (5.3.3)$$

This fuzzy shortest path networks can also be viewed in terms of the Dynamic programming (*DP)* recursion given in Equation (5.3.2). This recursion is very close to Ford's Algorithm and is easily extended to fuzzy numbers as in Equation (5.3.3). Then the DP recursion for lower fuzzy network is,

$$f_{iL}(S_{i+1}) = \min_{X_i} \{max[R_{iL}(X_i, S_{i+1}), f_{(i-1)L}(S_i)]\} \quad \rightarrow (5.3.4)$$

Where $f_{(i-1)L}(S_i)$ denotes the optimal value of the objective function corresponding to the last $i-1$ stages and S_i is the input to the stage $i-1$ of lower fuzzy networks$(FN)_L$, X_i denotes the vector of decision variable at stage i, $R_{iL}(X_i, S_{i+1})$ is the return function of the stage i and $f_{iL}(S_{i+1})$ denotes the optimal value of the objective function corresponding to the last i stages and S_{i+1} is the input to the stage i of lower fuzzy networks $(FN)_L$.

DP recursion for upper fuzzy network is,

$$f_{iU}(S_{i+1}) = \min_{X_i} \{max[R_{iU}(X_i, S_{i+1}), f_{(i-1)U}(S_i)]\} \quad \rightarrow (5.3.5)$$

Where $f_{(i-1)U}(S_i)$ denotes the optimal value of the objective function corresponding to the last $i-1$ stages and S_i is the input to the stage $i-1$ of upper fuzzy networks$(FN)_U$, X_i denotes the vector of decision variable at stage i, $R_{iU}(X_i, S_{i+1})$ is the return function of the stage i and $f_{iU}(S_{i+1})$ denotes the optimal value of the objective function corresponding to the last i stages and S_{i+1} is the input to the stage i of upper fuzzy networks$(FN)_L$.

Let us define *DP* recursion for Interval valued fuzzy network as,

$$f_{i-1}(S_i) = [f_{(i-1)L}(S_i), f_{(i-1)U}(S_i)] \quad \rightarrow (5.3.6)$$

Then by recursion

$$f_i(S_{i+1}) = [f_{iL}(S_{i+1}), f_{iU}(S_{i+1})] \quad \rightarrow (5.3.7)$$

By equation (5.3.4) and (5.3.5) we have,

$$f_i(S_{i+1}) = [\min_{X_i} \{max[R_{iL}(X_i, S_{i+1}), f_{(i-1)L}(S_i)]\},\ \min_{X_i} \{max[R_{iU}(X_i, S_{i+1}), f_{(i-1)U}(S_i)]\}]$$

$$= [\min_{X_i} \{max\{[R_{iL}(X_i, S_{i+1}), R_{iU}(X_i, S_{i+1})], [f_{(i-1)L}(S_i), f_{(i-1)U}(S_i)\}\}]$$

$$= [\min_{X_i} \{max[R_i(X_i, S_{i+1}), f_{i-1}(S_i)]\}] \quad (By\ (2.1.1)) \quad \rightarrow (5.3.8)$$

Where $f_i(S_{i+1})$ denotes the optimal value of the objective function corresponding to the last i stages and S_{i+1} is the input to the stage i of Interval valued fuzzy networks(*IVFN*), $f_{(i-1)}(S_i)$ denotes the optimal value of the objective function corresponding to the last $i-1$ stages and S_i is the input to the stage $i-1$ of Interval valued fuzzy networks*(IVFN)*, X_i denotes the vector of decision variable at stage i, $R_i(X_i, S_{i+1})$ is the return function of the stage i of Interval valued fuzzy networks(*IVFN*).

Definition 5.3.1

Shortest path in *IVFN* = Shortest path in lower limit fuzzy network $(FN)_L$
= Shortest path in upper limit fuzzy network $(FN)_{U.}$

Weight of the shortest path of IVFN = $[w_L, w_U]$ where w_L and w_U are weights of the fuzzy shortest path in $(FN)_L$ and $(FN)_U$ respectively

Now, we deal Interval valued fuzzy networks by using the algorithm [2], applied to $(FN)_L$ and $(FN)_U$ independently.

Algorithm 5.3.2

Step 1: Identify the decision variables and specify objective function to be optimized for interval valued fuzzy networks.

Step 2: Decompose the network into a number of smaller sub intervals. Identify the stage variable at each stage and write down the fuzzy transformation function as a function of the state variable and decision variable at the next stage.

Step 3: Write down a general recursive relationship for completing the fuzzy optimal policy of *IVFN* by using the interval valued fuzzy dynamic programming recursion in (5.3.6) and (5.3.8).

Step 4: Construct appropriate stage to show the required values of the return function at each Stage in *IVFN*.

Step 5: Determine the overall fuzzy optimal decision or policy and its value at each stage of an *IVFN*.

Step 6: We get the shortest path of *IVFN*.

Now, A_N^T be the interval valued fuzzy networks, representing the weight of N during time interval T.

$$A_N^T = [A_{NL}^T, A_{NU}^T] \rightarrow (5.3.9)$$

Where A_{NL}^T is the lower limit (R_{iL}) of the fuzzy network and A_{NU}^T is upper limit (R_{iU}) of the fuzzy network. Then,

$$\text{shortest path in } A_N^T = \text{ shortest path in } A_{NL}^T = \text{ shortest path in } A_{NU}^T \rightarrow (5.3.10)$$

$$\text{Weight of the shortest path of } IVFN = [\text{weight of the shortest path in } A_{NL}^T, \text{Weight of the shortest path in } A_{NU}^T] \rightarrow (5.3.11)$$

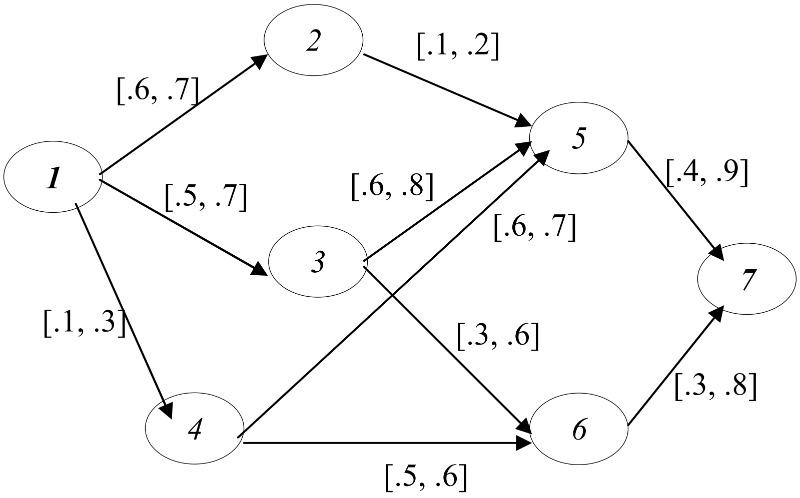
We shall illustrate the technique with a simple example and provide the mathematical verification.

Illustration 5.3.3

We consider a Network $N = (V, E)$ consisting n nodes (cities) and m edges (roads) connecting the cities of a country. If we measure the crowdness that is traffic of the roads of the network for particular time duration, it is quite impossible to measure the crowdness in duration as it is not fixed, but varies from time to time. So, appropriate technique to gradation of crowdness is an interval and not a point.

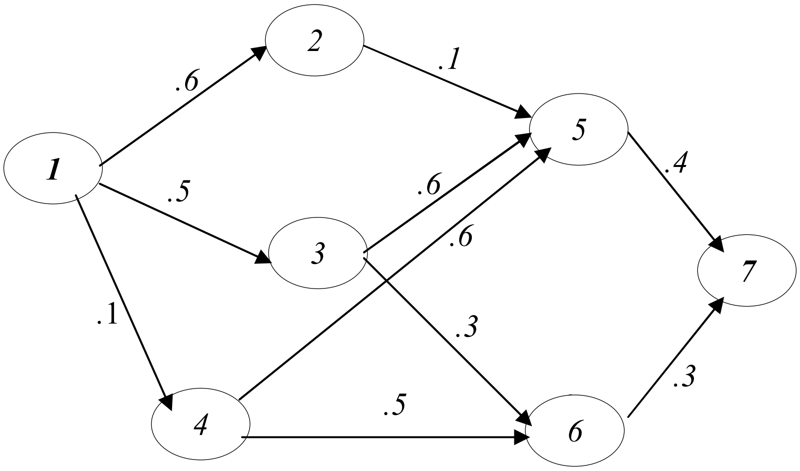
In this case, the network N is an interval valued fuzzy network in which the weight of the each arc $(i, i+1)$ depends upon the crowdness.

Suppose that we want to select the shortest highway route (path) between two cities. The following route network provides the possible routes between the starting city at node 1 and the destination city at node 7. The routes pass through intermediate cities designated by nodes 2 to 6.



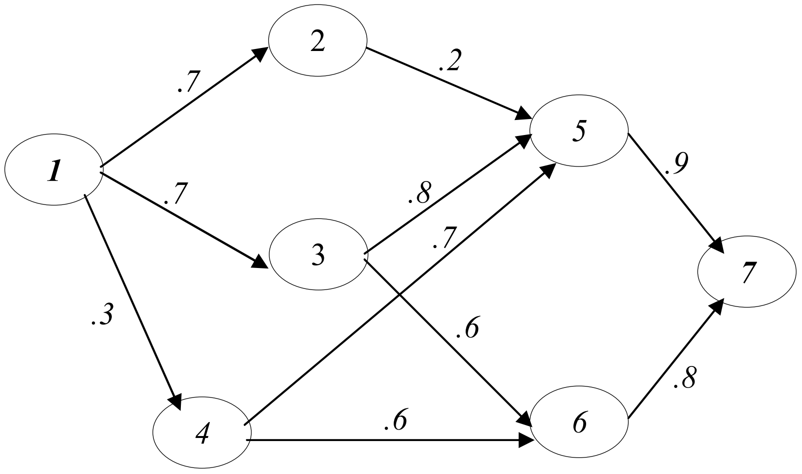
A_N^T- Internal valued fuzzy network *(IVFN)*

By using our representation (5.3.1) and (5.3.11), $A_N^T = [\, A_{NL}^T,\ A_{NU}^T]$



and

A_{NL}^T - lower limit fuzzy network *(FN)*$_L$

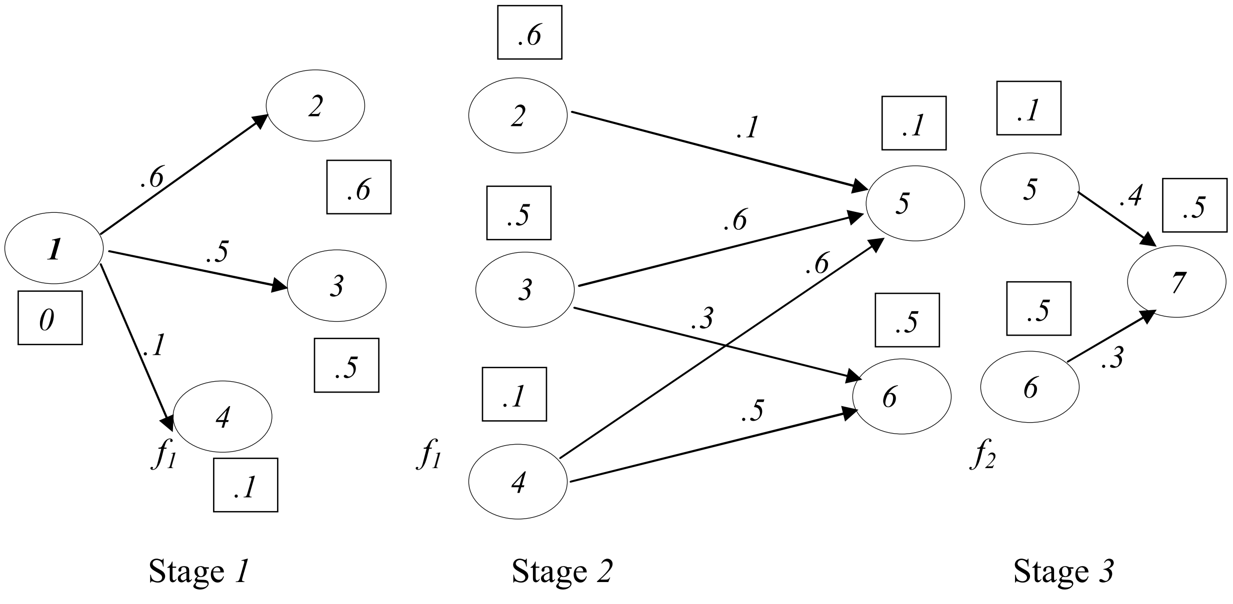


A_{NU}^{T} - lower limit fuzzy network $(FN)_U$

Now we apply the algorithm (5.3.2) to find a path between city *1* to city *7* which is minimum among all the paths between city 1to city *7*.

(i) Shortest path for the lower limit fuzzy network.

First decompose the lower limit fuzzy network into sub networks/stages as



Now S_1 is the state in which the node 1lies also, S_1 has only state value $S_1=1$. State S_2 has only three possible values; Say *2, 3, 4* corresponding stage *1*, and so on. Possible alternative paths from one stage to the other will be called decision variables by X_i the decision which takes from S_{i-1} to S_i. The return or the gain which obviously being the function of decision will be denoted by $R_{iL}(X_i,S_{i+1})$. Here $R_{iL}(X_i, S_{i+1})$ can be identified with the lower limit of the corresponding arc.

By equation (5.3.4) we have,

$$f_{iL}(S_{i+1}) = \min_{X_i}\{\max[R_{iL}(X_i, S_{i+1}), f_{(i-1)L}(S_i)]\}$$

Now initially for $i = 0,\ f_i(S_{i+1}) = f_0(S_1) = f_0(1) = 0$

For $i = 1$ (stage*1*):

$$f_1(S_2) = \min_{X_i}\{\max[R_{iL}(X_1, S_2), f_0(S_1)]\}$$

$$= \min_{X_i}[R_{iL}(X_1, S_2)]$$

Now tabulating the date for $f_1(S_2)$

S_1	S_2	X_1	$R_{iL}(X_1, S_2)$	$f_1(S_2)$	fuzzy optimal policy
	2	*1- 2*	*.6*	*.6*	*1- 2*
1	*3*	*1- 3*	*.5*	*.5*	*1- 3*
	4	*1- 4*	*.1*	*.1*	*1- 4*

For stage *2 (i = 2)*

$$f_2(S_3) = \min_{X_2}\{\max[R_{2L}(X_2, S_3), f_1(S_2)]\}$$

S_2	S_3	X_2	$R_{2L}(X_2,S_3)$	$max\ (R_2,f_1)$	$f_2\ (S_3)$	fuzzy optimal policy
2	5 6	2-5	.1	.6		2-5
		3-5 3-6	.6 .3	.6 .5*	.5	3-5
3 4		4-5 4-6	.6 .5	.6 .5*	.5	4- 6

For last stage $3\ (i = 3)$

$$f_3(S_4) = \min_{X_3} \{max\ [R_{3L}\ (X_3, S_4),\ f_2\ (S_3)]\}$$

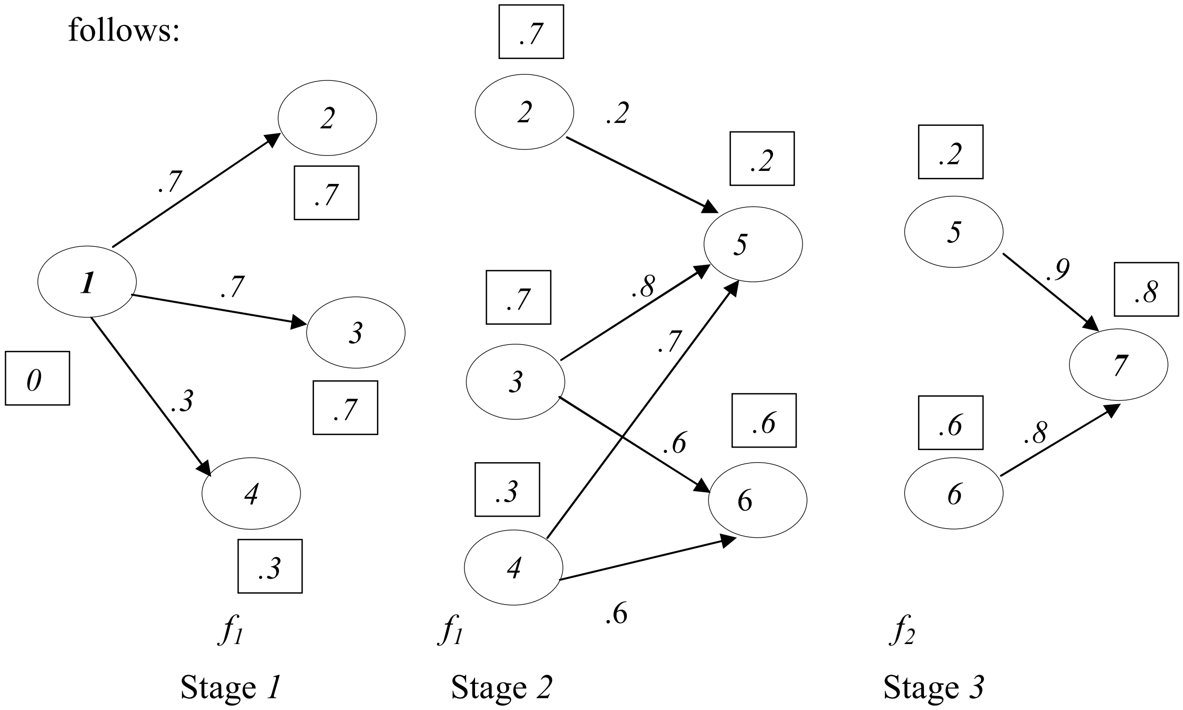
S_3	S_4	X_3	$R_{3L}\ (X_3,S_4)$	$max\ (R_3, f_2)$	f_3	fuzzy optimal policy
5	7	5-7	.4	.6 .6 .6		
6		6-7	.3	.5* .5	.5	6-7

Therefore, for the lower limit fuzzy network of the shortest path from city 1to city 7 is: $1 \longrightarrow 4 \longrightarrow 6 \longrightarrow 7$

Weight of the shortest path $W_L = (.1, .5, .3)$.

(ii) Shortest path for the upper limit fuzzy matrices

Decompose the upper limit fuzzy network in to sub network /stages as follows:



Similarly we have to find the upper limit of the shortest path. Here $R_{iU}(X_i, S_i+1)$ can be identified with the upper limit of the corresponding arc.

By equation (5.3.5) we have,

$$f_{iU}(S_i+1) = \min_{X_i} \{\max[R_{iU}(X_i, S_{i+1}), f_{(i-1)U}(S_i)]\}$$

Now, initially for $i = 0, f_i(S_{i+1}) = f_0(S_1) = f(1) = 0$

For $i = 1$ (stage 1):

$$f_1(S_2) = \min_{X_1}\{\max[R_{1U}(X_1, S_2), f_0(S_1)]\}$$

$$= \min_{X_1}[R_{1U}(X_1, S_2)]$$

Now tabulating the data for $f_1(S_2)$

S_1	S_2	X_1	$R_{1U}(X_1, S_2)$	$f_1(S_2)$	fuzzy optimal policy
1	2	1- 2	.7	.7	1- 2
	3	1- 3	.7	.7	1- 3
	4	1- 4	.3*	3*	1- 4

For stage 2 $(i = 2)$:

$$f_2(S_3) = \min_{X_2}\{\max [R_2(X_2, S_3),\ f_1(S_2)]\}$$

S_2	S_3	$R_{2U}(X_2,S_3)$	$\max(R_2, f_1)$	$f_2(S_3)$	fuzzy optimal policy
2		2-5	.7		2-5
3		3-5	.8	.7	3-6
	5	3-6	.7*		
	6	4-5	.7		
4		4-6	.6*	.6	4-6

For last stage 3 $(i = 3)$

$$f_3(S_4) = \min_{X_3}\{\max [R_{3U}(X_3,S_4),\ f_2(S_3)]\}$$

S_3	S_4	X_3	$R_{3L}\ (X_3,S_4)$	$max\ (R_3,f_2)$	F_3	fuzzy optimal policy
5	7	5-7	.9	.9 .9 .9		
6		6-7	.8	.8* .8	.8	6-7

Therefore the shortest path from city 1 to city 7 for the upper limit fuzzy network is: $1 \longrightarrow 4 \longrightarrow 6 \longrightarrow 7$

Weight of the shortest path $w_U = (.3, .6, .8)$

Now we conclude by equation (5.3.10) we have,

Shortest path in $A_N{}^T$ = shortest path in $A_{NL}{}^T$ = shortest path in $A_{NU}{}^T$

$$= 1 \longrightarrow 4 \longrightarrow 6 \longrightarrow 7$$

By equation (5.3.11) we have,

Weight of the shortest path of *IVFN* = [weight of the shortest path in $A_{NL}{}^T$, weight of the shortest path in $A_{NU}{}^T$]

That is, $w = [w_L\, , w_U]$

$$= [(.1, .5, .3),\quad (.3, .6, .8)]$$

$$= [.1, .3\ ], [.5, .6], [.3, .8]$$

Therefore shortest path of *IVFN* is $1 \xrightarrow{[.1,\ .3\ ]} 4 \xrightarrow{[.5,\ .6]} 6 \xrightarrow{[.3,\ .8]} 7.$

****

CONCLUSION AND SCOPE

We have represented an *IVFM* as an interval matrix of two fuzzy matrices. We have defined regular interval valued Fuzzy Matrices as a generalization of regular Fuzzy Matrices and the structure of Row space and Column space of an IVFM are obtained. This leads to a characterization of regular IVFM and Invertible IVFM. We have discussed the g-Inverses of Interval Valued Fuzzy Matrices (IVFM) as a generalization of g- inverses of regular fuzzy matrices. The existence and construction of g-inverses, $\{1, 2\}$ inverses, $\{1, 3\}$ inverses and $\{1, 4\}$ inverses of Interval valued fuzzy matrix are determined in terms of the row and column spaces. There is a scope for further study on spectral inverses such as group inverse, Drazin inverse and the spectral properties of interval valued fuzzy matrices.

We have discussed the consistency of the Interval valued fuzzy relational equations as a generalization of fuzzy relational equations and determine the complete set of solutions of $xA = b$ where A is an *IVFM* and b is an IVF Vector, equivalent condition for the existence of Interval maximum solution of $xA = b$ for $A \in$IVFM and its relation with the maximum solution of the System $xA^I = b^I$ is determined, where $A^I = \{ A'/ A_L \leq A' \leq A_U\}$ and $b^I = \{ b'/ b_L \leq b' \leq b_U\}$. Katarina Cechlarova deals with the situation when the right-hand side of an unsolvable max-min fuzzy system remains constant and allowed to modify the matrix to get solvable systems. There is a scope for further study on unsolvable systems of interval valued fuzzy (max-min) relational equations.

As an application, we have highlighted the role of *IVFM* in Document Retrieval System, Medical Diagnosis and Shortest path Network with suitable illustration. There is a scope for further study on Application of interval valued fuzzy matrices in other areas such as agriculture, economics, engineering and environment.

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1. **AR. Meenakshi, M. Kaliraja**, *Regular Interval valued Fuzzy matrices,* Advances in Fuzzy mathematics, Vol.5 (1) (2010) 7-15.
2. **AR. Meenakshi**, **M. Kaliraja**, *On Regularity of Interval valued Block Fuzzy Matrices*, Advances in Algebra, Vol.3 (1) (2010) 7–14.
3. **AR. Meenakshi**, **M. Kaliraja**, *Interval Valued Fuzzy Relational Equations*, International Journal of Computational Cognition (To Appear).
4. **AR. Meenakshi, M. Kaliraja**, *g- Inverses of Interval Valued Fuzzy Matrices*, ITB Journal of Science (Accepted).
5. **AR. Meenakshi**, **M. Kaliraja**, *An Application of Interval Valued Fuzzy Matrices in Medical Diagnosis*, International Journal of Mathematical Analysis (Accepted).
6. **AR. Meenakshi, M. Kaliraja**, *Transitive Closure of an Interval Valued Fuzzy Matrix and its Application in Document Retrieval Systems*, (Communicated).
7. **AR. Meenakshi**, **M. Kaliraja**, *Determination of the Shortest Path in Interval Valued Fuzzy Networks,* (Communicated).

Papers Presented in the Conference:

1. **M. Kaliraja,** *Regular Interval valued Fuzzy matrices,* Paper Presented in the "1st Annual Research congress (KUARC)", 7th – 10th, December 2009, Karpagam University, Tamil Nadu.
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3. **M. Kaliraja,** *Solutions of Interval Valued Fuzzy Relational Equations*; Paper Presented in the National Conference on Recent Trends in Fuzzy Mathematics, June 14th – 15th, 2010, Annamalai University, Tamil Nadu.

4 **M. Kaliraja,** *Transitive Closure of an Interval Valued Fuzzy Matrix and its Application in Document Retrieval Systems*, Paper Presented in the "2nd Annual Research congress (KUARC)", 8th – 11th, December 2010, Karpagam University, Coimbatore, Tamil Nadu.

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